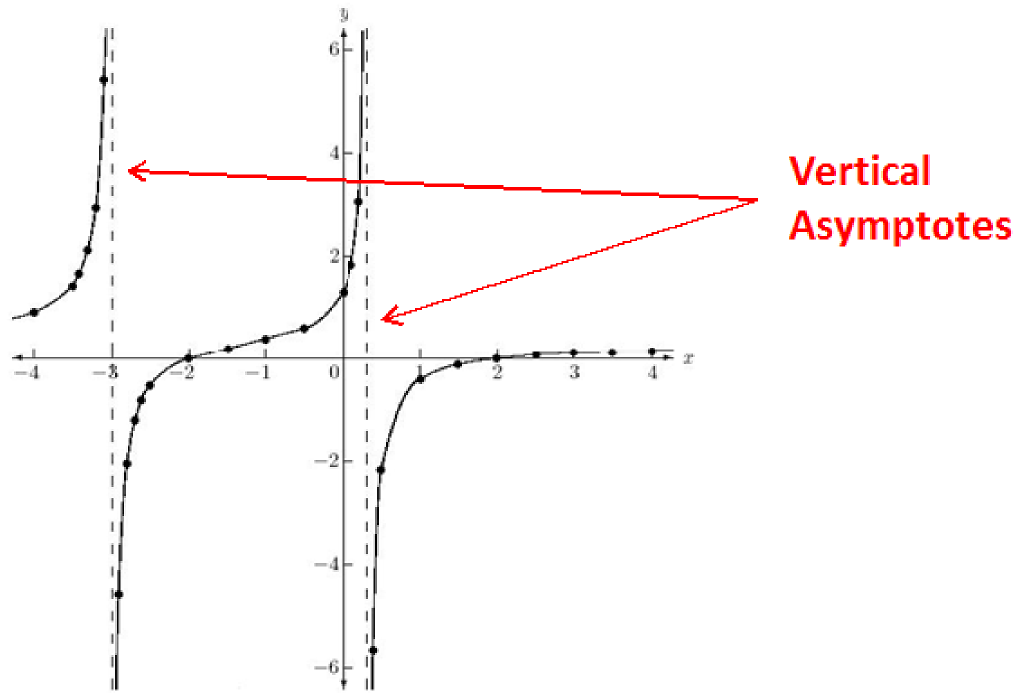
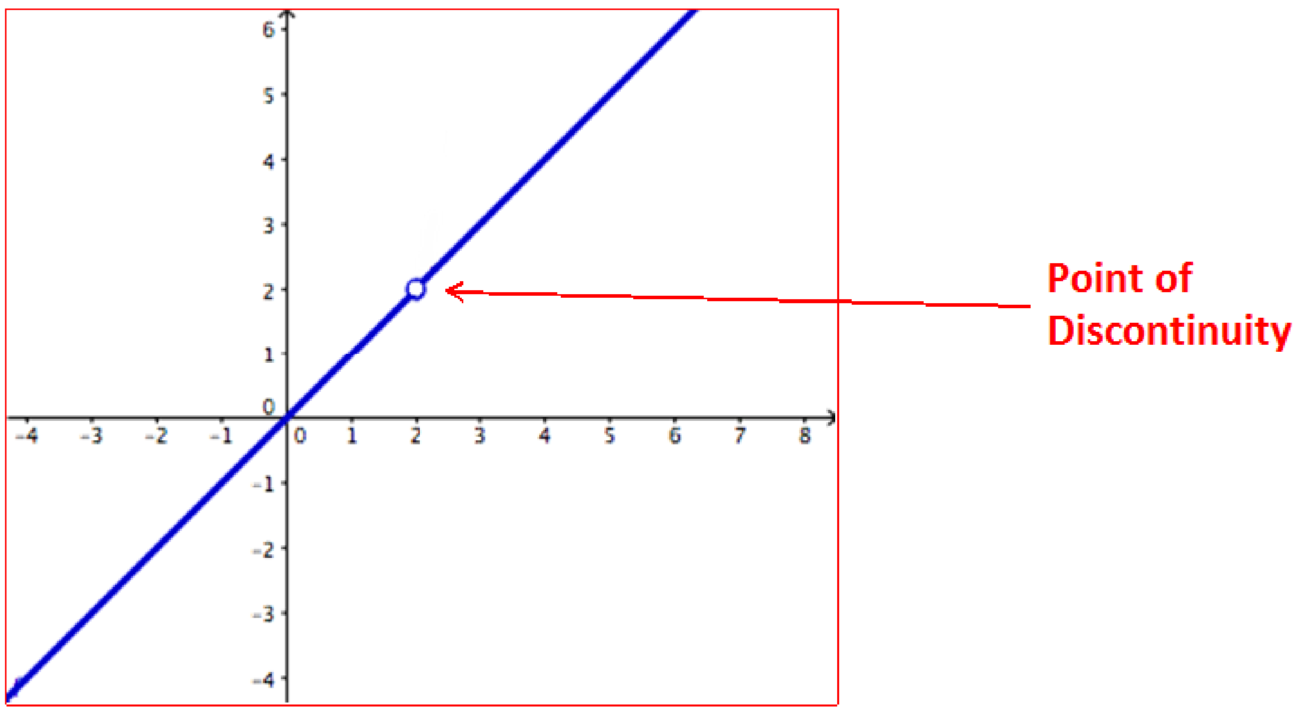
9.2 Analysing Rational Functions

Graphs of rational functions can have a variety of shapes and different features—vertical asymptotes are one such feature. A vertical asymptote of the graph of a rational function corresponds to a non-permissible value in the equation of the function, but not all non-permissible values result in vertical asymptotes. Sometimes a non-permissible value instead results in a **point of discontinuity** in the graph.

holes

POD




Point of
Discontinuity

Ex.1 Sketch the graph of the function $f(x) = \frac{x^2-x-6}{x+2}$.
Analyse its behaviour near its non-permissible values

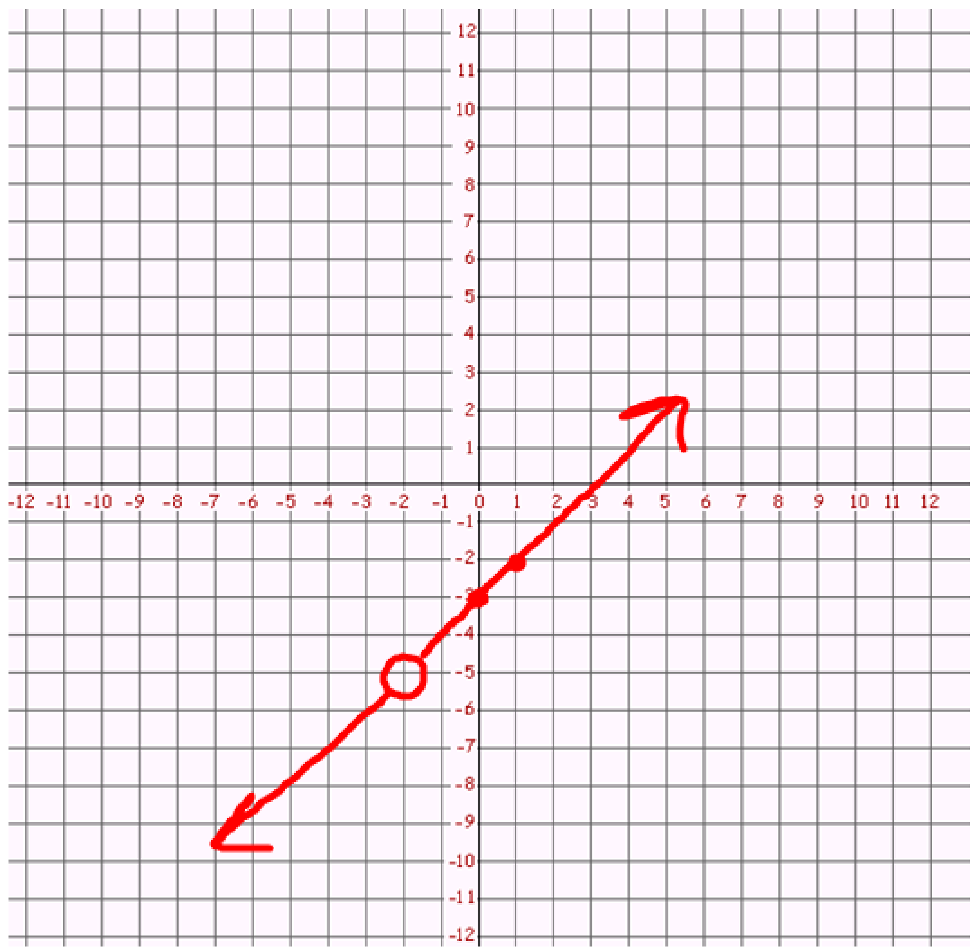
$$f(x) = \frac{(x-3)\cancel{(x+2)}}{\cancel{(x+2)}}$$

NPV

$x = -2$

$$\boxed{f(x) = x-3}$$

x	y
-2	-5
1	-2
0	-3



- **Points of Discontinuity: At Least One Common Factor**

  If the numerator and denominator have at least one common factor, there is at least one point of discontinuity in the graph.

  – Equate the common factor(s) to zero and solve for x to determine the x-coordinate of the point of discontinuity.

  – Substitute the x-value in the simplified expression to find the y-coordinate of the point of discontinuity.

Example: $y = \frac{(x-4)(x+2)}{x+2}$

$x + 2 = 0$: the x-coordinate of the point of discontinuity is -2.

Substitute $x = -2$ into the simplified equation:

$y = x - 4$

$y = \mathbf{-2} - 4$

$y = -6$

point of discontinuity: $(-2, -6)$

Your Turn

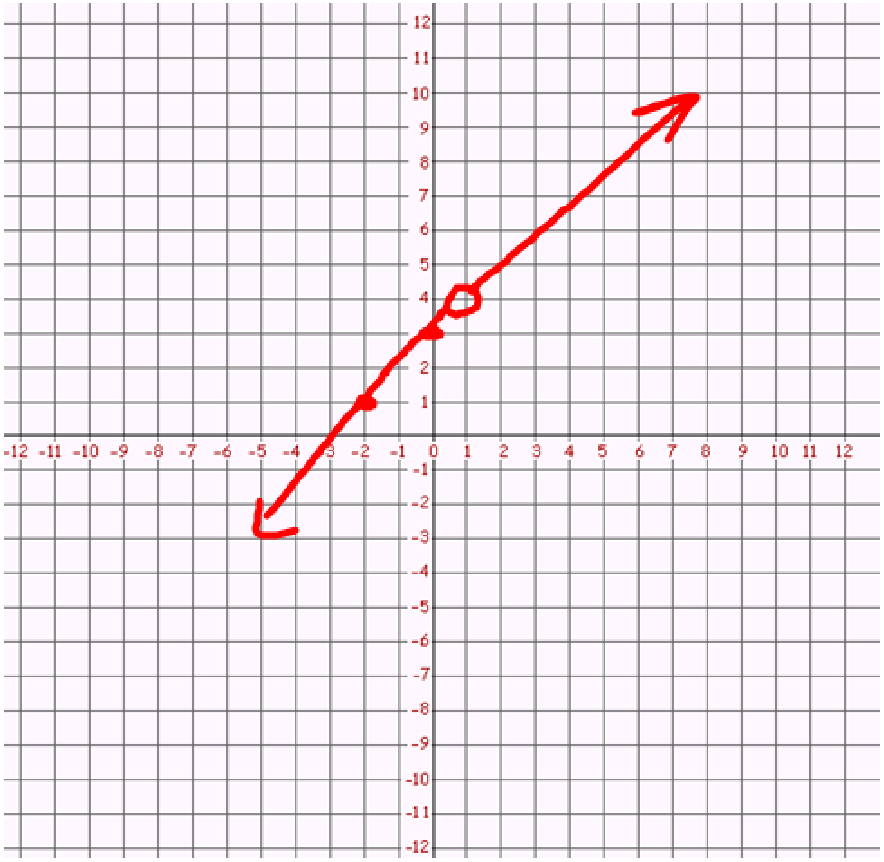
Sketch the graph of the function $f(x) = \dfrac{x^2 + 2x - 3}{x - 1}$. Analyse its behaviour near its non-permissible value.

$$f(x) = \frac{(x+3)\cancel{(x-1)}}{\cancel{(x-1)}}$$

NPV $x \neq 1$

$$f(x) = x + 3$$

x	y
1	4
0	3
-2	1



Working Example 1: Graph a Rational Function With a Point of Discontinuity

Sketch the graph of $f(x) = \frac{x^2 - 3x - 4}{x - 4}$.

Solution

Fully factor the numerator and denominator of the rational function.

$$f(x) = \frac{\cancel{(x-4)}(x+1)}{\cancel{(x-4)}}$$

There is a common factor, so the graph of the function has a POD.

Simplify the rational function. What type of equation remains after the function is simplified?

$f(x) = x + 1$ Linear

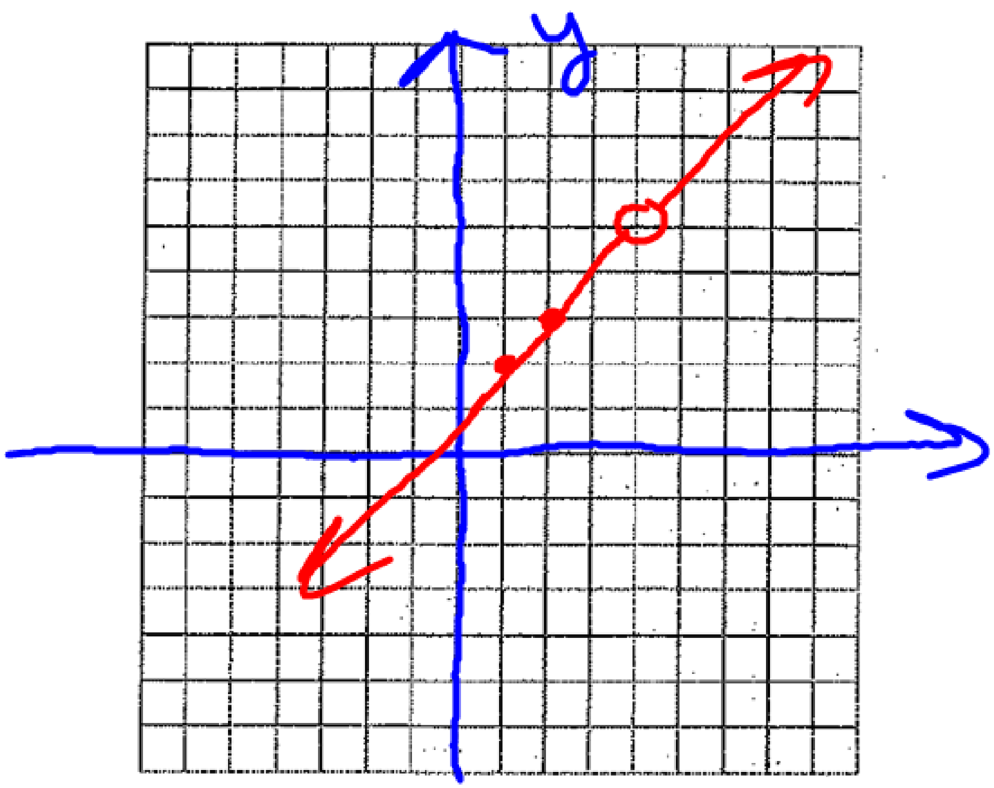
Equate the common factor to zero and solve for x. Doing so identifies the X-value of the POD

Substitute the value of x into the simplified function and solve for y. Doing so identifies the

y -value of the point of discontinuity in the graph.

The point of discontinuity is (4, 5).

Graph the rational function, labelling the point of discontinuity.



$f(x) = x + 1$

x	y
4	5
1	2
2	3

Working Example 3: Sketch a Discontinuous Rational Function

Sketch the graph of $f(x) = \frac{x^2 + 2x - 8}{x^2 + 5x + 4}$. Label all important parts of the graph.

Solution

Fully factor the numerator and denominator of the rational function. Simplify the rational function.

$$f(x) = \frac{\cancel{(x+4)}(x-2)}{\cancel{(x+4)}(x+1)}$$

NPV

$x \neq -4, -1$

Equate the common factor to zero and solve for x to establish the x coord. of the point of discontinuity. Substitute the value of x into the simplified function and solve for y to establish the y coord of the point of discontinuity in the graph.

The point of discontinuity is at $(-4, 2)$.

Find the x-intercept and y-intercept of the simplified function.

$$f(x) = \frac{x-2}{x+1}$$

Find the horizontal and vertical asymptotes of the simplified function. Then, use the information you have generated to graph the general shape of the rational function. Label the point of discontinuity, the asymptotes, and the intercepts. Check your sketch using technology.

$$f(x) = \frac{x-2}{x+1}$$

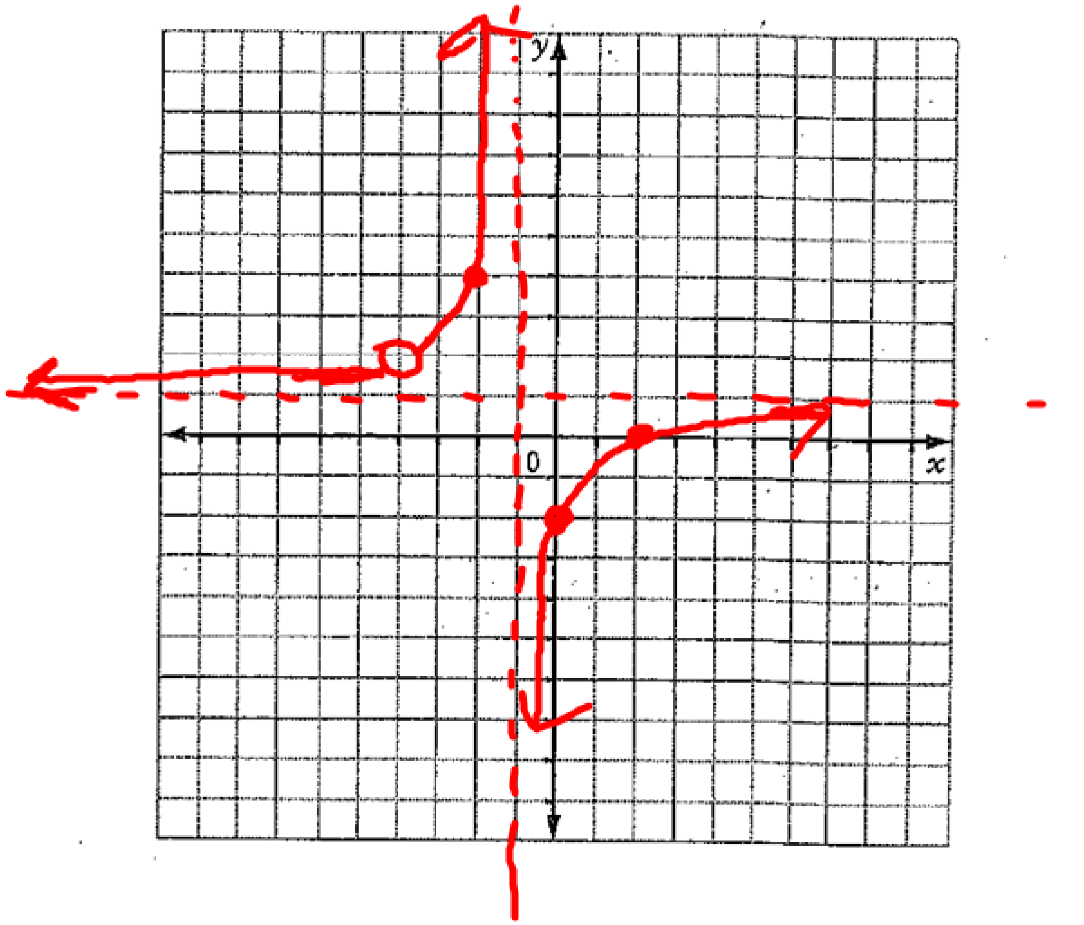
<u>VA</u>

$x = -1$

<u>HA</u>

$y = 1$

x	y
-2	4
0	-2
2	0



Working Example 2: Compare Points of Discontinuity and Asymptotes in Rational Functions

Compare the graphs of $f(x) = \frac{x^2 + 4x + 3}{x + 1}$ and $g(x) = \frac{x^2 - 4x + 3}{x + 1}$.

Fully factor the numerator and denominator of each rational function. Simplify the rational functions, if possible. How do the two simplified equations differ?

$$f(x) = \frac{(x+3)\cancel{(x+1)}}{\cancel{(x+1)}}$$

$$g(x) = \frac{(x-3)(x-1)}{(x+1)}$$

$$f(x) = x + 3$$

When simplified, $f(x)$ is a linear function. It has a(n) POD.
(*point of discontinuity* or *asymptote*)

When simplified, $g(x)$ is a rational function. It has a(n) Asymptote.
(*point of discontinuity* or *asymptote*)

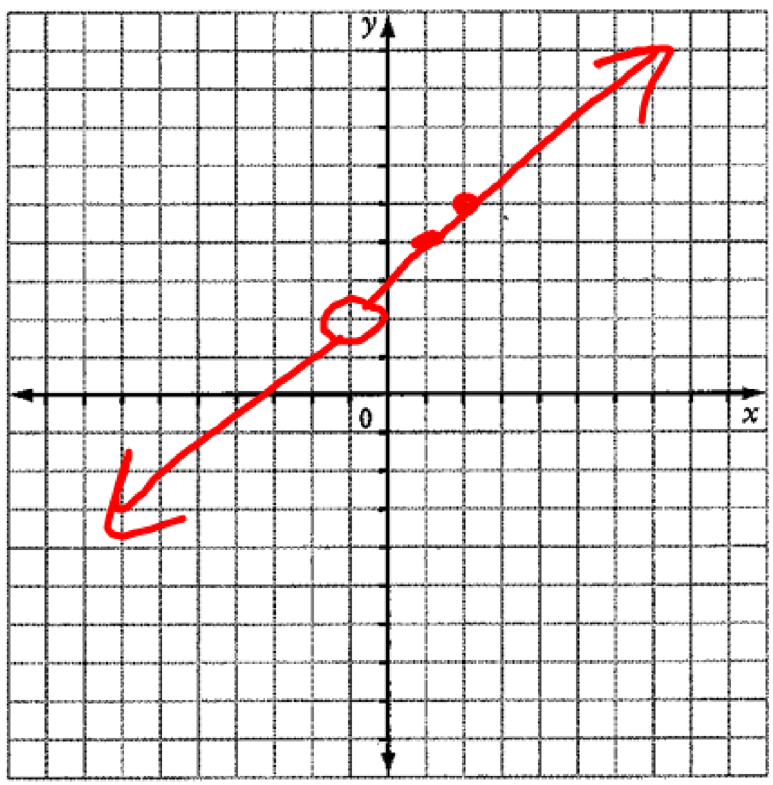
x	y
-1	2
1	4
2	5

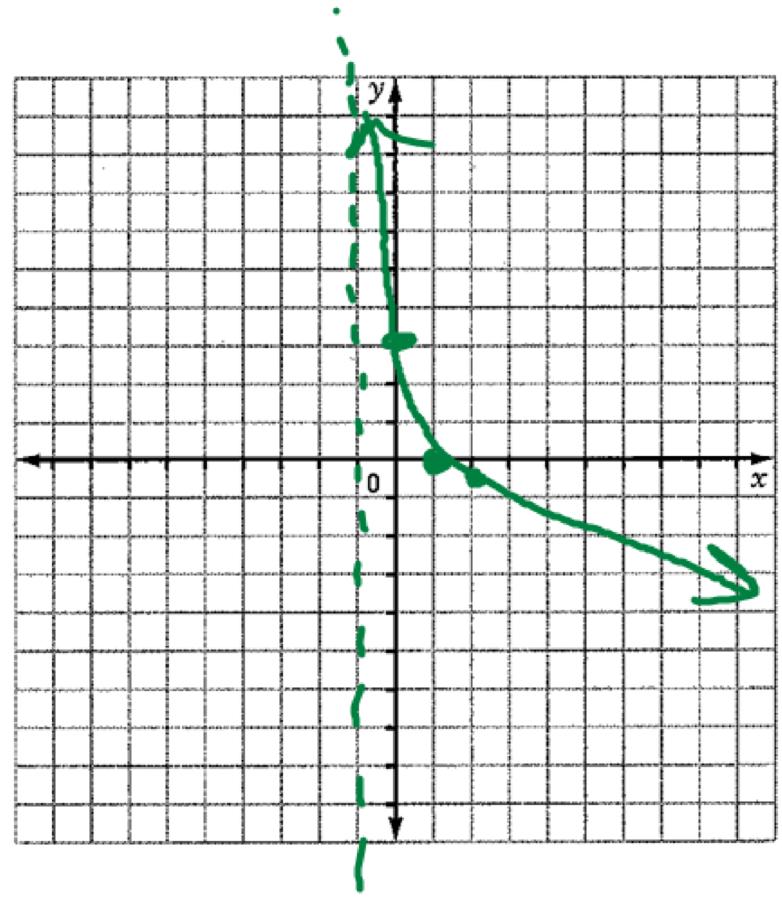
Sketch $f(x)$ and $g(x)$ below

$$f(x) = \frac{x^2 + 4x + 3}{x + 1}$$

x	y
1	0
2	-1/3
0	3

$$g(x) = \frac{x^2 - 4x + 3}{x + 1}.$$





Your Turn

Compare the functions $f(x) = \frac{x^2 - 3x}{2x + 6}$ and $g(x) = \frac{x^2 + 3x}{2x + 6}$ and explain any differences.

$$f(x) = \frac{x(x-3)}{2(x+3)}$$

$$\frac{\text{NPV}}{x \neq -3}$$

VA

$$g(x) = \frac{x\cancel{(x+3)}}{2\cancel{(x+3)}}$$

$$\frac{\text{NPV}}{x \neq -3}$$ POD

$$\left(-3, -\tfrac{3}{2}\right)$$

Verify your results by graphing with technology.

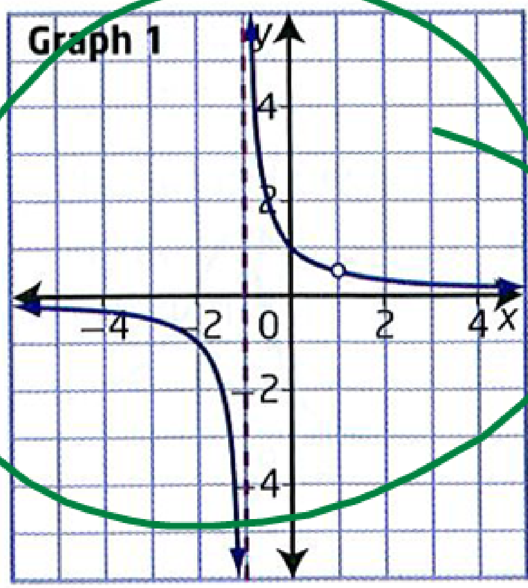
$\frac{x^2+2}{(x-2)(x+1)}$ $\frac{\cancel{x-1}}{\cancel{(x-1)}(x+1)}$ $\frac{1}{x+1}$

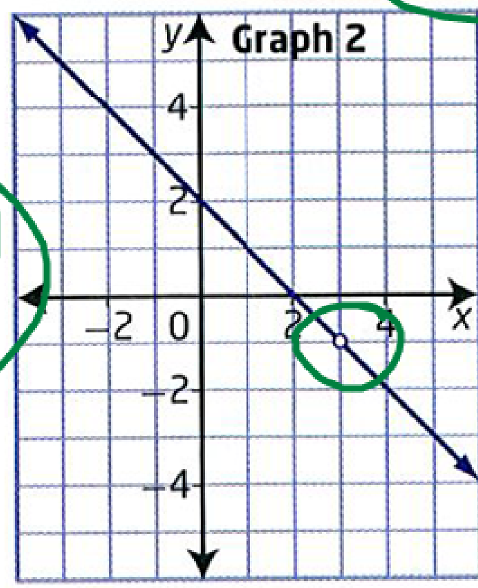
Your Turn

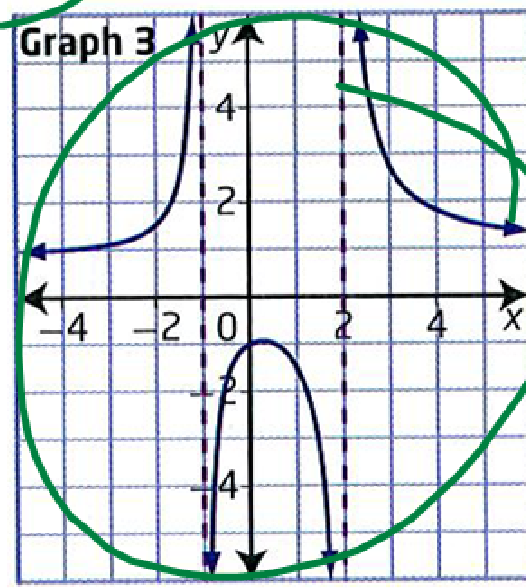
Match the equation of each rational function with the most appropriate graph. Explain your reasoning.

$-\cancel{(x-3)}(x-2)$

3 $K(x) = \frac{x^2+2}{x^2-x-2}$ 1 $L(x) = \frac{x-1}{x^2-1}$ 2 $M(x) = \frac{x^2-5x+6}{\cancel{3-x}}$







Key Ideas

- The graph of a rational function
    - has either a vertical asymptote or a point of discontinuity corresponding to each of its non-permissible values
    - has no vertical asymptotes or points of discontinuity
- To find any x-intercepts, points of discontinuity, and vertical asymptotes of a rational function, analyse the numerator and denominator.
    - A factor of only the numerator corresponds to an x-intercept.
    - A factor of only the denominator corresponds to a vertical asymptote.
    - A factor of both the numerator and the denominator corresponds to a point of discontinuity.
- To analyse the behaviour of a function near a non-permissible value, use a table of values or the graph, even though the function is undefined or does not exist at the non-permissible value itself.

Assignment
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#'s 1 ,2 (just determine if there is a VA or point of discontinuity)
#4, don't sketch, 5,6,7,8,9a