6.3 Optimization Problems P. 277 1-27

1. b) $s(x) = x(15-x)$ a) $7 \cdot 8 = 56$

$s(x) = 15x - x^2$

$s'(x) = 15 - 2x$

$0 = 15 - 2x$

$2x = 15$

$\therefore x = 7.5$

c) $(7.5)(7.5) = 56.25$

d) b includes all real #'s, not just integers

2. $s(x) = x(x+6)$

$s(x) = x^2 + 6x$

$s'(x) = 2x + 6$

$0 = 2(x+3)$

$x = -3$

$x+6 = 3$

product $\rightarrow$ -9

3. $s(x) = x^2 + (x+10)^2$

$s(x) = x^2 + x^2 + 20x + 100$

$s(x) = 2x^2 + 20x + 100$

$s'(x) = 4x + 20$

$0 = 4(x+5)$

$x = -5$

$x+10 = 5$

$(-5)^2 + (5)^2$

$25 + 25 = 50$

4. $s(x) = (x)^2 + \left(\frac{16}{x}\right)^2$ $xy = 16$

$s(x) = x^2 + \frac{256}{x^2} = x^2 - 256x^{-2}$ $y = \frac{16}{x}$

$s'(x) = 2x - 512x^{-3}$

$s'(x) = 2x^{-3}(x^4 - 256)$

$= 2x^{-3}(x^2-16)(x^2+16)$

$= 2x^{-3}(x+4)(x-4)(x^2+16)$

$x = 0$ $x = -4$ $x = 4$

$(-4)^2 + \left(\frac{16}{-4}\right)^2$

$16 + 16 = 32$

6.3 Optimization Problems

5. $s(x) = x(50-x)$

$s(x) = 50x - x^2$

$s'(x) = 50 - 2x$

$0 = 2(25-x)$

$x = 25$ $\quad 25(25) = 625$

$50 - x = 25$

6. $s(x) = x - \sqrt{x}$ $\qquad x^{1/2}$

$s'(x) = 1 - \frac{1}{2}x^{-1/2}$

$0 = 1 - \frac{1}{2}x^{-1/2}$

$\frac{1}{2}x^{-1/2} = 1$

$x^{-1/2(-2)} = 2^{(-2)}$

$x = 2^{-2}$

$x = \frac{1}{4}$

7. $s(x) = (21-x)(x)^2$

$s(x) = 21x^2 - x^3$

$s'(x) = 42x - 3x^2$

$0 = 3x(14-x)$

$x = 0 \quad x = 14$ $\qquad 7 \times (14)^2 = 1372$

$21 - 14 = 7$

8. $s(x) = (20-x)^2(x^3)$

$= (400 - 40x + x^2)x^3$

$= 400x^3 - 40x^4 + x^5$

$s'(x) = 1200x^2 - 160x^3 + 5x^4$

$s'(x) = 5x^2(240 - 32x + x^2)$

$0 = 5x^2(x-20)(x-12)$

$= x = 0, \quad x = 20, \quad x = 12$

reject reject $\quad 20 - 12 = 8$

Check:

$(12)^2(8)^3$

$(144)(512)$

$= 73728$

$(8)^2(12)^3$

$64(1728)$

$= 110592$ ← larger

$(12^3)(8)^2 = 110592$

The numbers are 12 & 8 (where 12 is cubed)

6.3 Optimization Problems

9. $S(x) = (x) + \frac{100}{x}$

$S'(x) = 1 - 100x^{-2}$

$0 = x^{-2}(x^2 - 100)$

$0 = x^{-2}(x+10)(x-10)$

$x = 0 \quad x = -10 \quad x = 10$

Minimum sum

$10 + 10 = 20$

10. $S(x) = \frac{1}{x} + \frac{1}{10-x}$

$S(x) = x^{-1} + (10-x)^{-1}$

$S'(x) = -x^{-2} - 1(10-x)^{-2}(-1)$

$S'(x) = -x^{-2} + (10-x)^{-2}$

$S'(x) = -\frac{1}{x^2} + \frac{1}{(10-x)^2}$

$S'(x) = \frac{-(10-x)^2 + x^2}{(x^2)(10-x)^2}$

$S'(x) = \frac{-100 + 20x - x^2 + x^2}{(x^2)(10-x)^2}$

$\frac{0}{1} = \frac{-100 + 20x}{(x^2)(10-x^2)}$

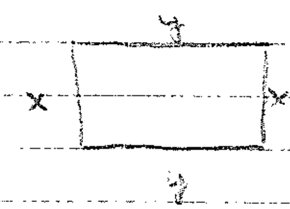
$0 = -20(5-x)$

$x = 5$

Sum: $\frac{1}{5} + \frac{1}{5}$

$= \frac{2}{5}$

11. Perimeter = 400m



$2x + 2y = 400$

$x + y = 200$

$y = 200 - x$

Area $\Rightarrow x \cdot y$

$S(x) = x(200 - x)$

$S(x) = 200x - x^2$

$S'(x) = 200 - 2x$

$S'(x) = 2(100 - x)$

$0 = 2(100 - x)$

$x = 100$

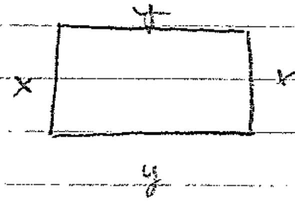
Area $= 100(200 - 100)$

$= 100(100)$

$= 10000\ m^2$

6.3- Continued

12. Area = $400 m^2$



$xy = 400$

$y = \frac{400}{x}$

Perimeter $\Rightarrow 2x + 2y$

$s(x) = 2x + 2\left(\frac{400}{x}\right)$

$s(x) = 2x + 800x^{-1}$

$s'(x) = 2 - 800x^{-2}$

$s'(x) = 2 - \frac{800}{x^2}$

$s'(x) = \frac{2x^2 - 800}{x^2}$

$0 = \frac{2(x^2 - 400)}{x^2}$

$0 = 2(x-20)(x+20)$

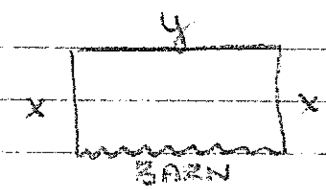
$x = 20$ $\quad$ $x = -20$ reject

Min. Perimeter

$2(20) + 2(20)$

$= 40 + 40$

$= 80m$

13. 

$x + x + y = 32$

$y = 32 - 2x$

Area $\Rightarrow x \cdot y$

$s(x) = x(32 - 2x)$

$s(x) = 32x - 2x^2$

$s'(x) = 32 - 4x$

$0 = 4(8 - x)$

$x = 8$

Max Area

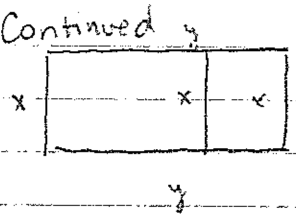
$x \cdot y$

$= 8(32 - 2(8))$

$= 8(16)$

$= 128 m^2$

6.3 Continued

14. 

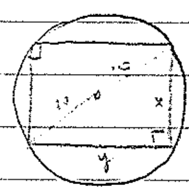
$$3x + 2y = 600$$
$$2y = 600 - 3x$$
$$y = 300 - \frac{3}{2}x$$

Area $\rightarrow x \cdot y$

$$s(x) = x\left(300 - \frac{3}{2}x\right)$$
$$s(x) = 300x - \frac{3}{2}x^2$$
$$s'(x) = 300 - 3x$$
$$0 = 3(100 - x)$$
$$x = 100$$
$$y = 300 - \frac{3}{2}(100)$$
$$y = 300 - 150$$
$$y = 150$$

Area $\rightarrow 100(150)$

$= 15000\ m^2$

15. 

pythagorean theorem

$$x^2 + y^2 = r^2$$
$$x^2 + y^2 = 20^2$$
$$x^2 + y^2 = 400$$
$$y = \sqrt{400 - x^2}$$

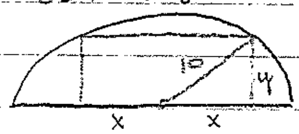
rectangle area $\rightarrow xy$

$$s(x) = x\left(\sqrt{400 - x^2}\right)$$
$$s(x) = x(400 - x^2)^{1/2}$$
$$s'(x) = (1)(400 - x^2)^{1/2} + (x)\left(\tfrac{1}{2}\right)(400 - x^2)^{-1/2}(-2x)$$
$$s'(x) = (400 - x^2)^{1/2} - x^2(400 - x^2)^{-1/2}$$
$$s'(x) = (400 - x^2)^{-1/2}\left[400 - x^2 - x^2\right]$$
$$s'(x) = (400 - x^2)^{-1/2}(400 - 2x^2)$$
$$s'(x) = \frac{2(200 - x^2)}{(400 - x^2)^{1/2}}$$
$$0 = 2(10\sqrt{2} + x)(10\sqrt{2} - x)$$
$$x = -10\sqrt{2} \quad x = 10\sqrt{2}$$
$$y = 10\sqrt{2}$$

The dimensions are $10\sqrt{2} \times 10\sqrt{2}$

6.3- Continued

16



pythagorean theorem: $x^2 + y^2 = r^2$

$x^2 + y^2 = 10^2$

$y = \sqrt{100 - x^2}$

Area → $2x(y)$

$S(x) = 2x\left(\sqrt{100-x^2}\right)$

$S'(x) = (2)(100-x^2)^{1/2} + (2x)(\tfrac{1}{2})(100-x^2)^{-1/2}(-2x)$

$S'(x) = 2(100-x^2)^{1/2} - 2x^2(100-x^2)^{-1/2}$

$S'(x) = 2(100-x^2)^{-1/2}\left[100 - x^2 - x^2\right]$

$S'(x) = 2(100-x^2)^{-1/2}\left[100 - 2x^2\right]$

$0 = \dfrac{4(50-x^2)}{(100-x^2)^{-1/2}}$

$0 = 4(\sqrt{50}-x)(\sqrt{50}+x)$

$x = \sqrt{50}$ $\quad$ $x = -\sqrt{50}$

$x = 5\sqrt{2}$ $\quad$ $x = -5\sqrt{2}$

$y = \sqrt{100-50}$

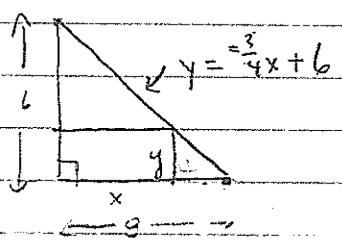
$y = 5\sqrt{2}$

Length → $5\sqrt{2} + 5\sqrt{2}$

$= 10\sqrt{2}$ cm

Width → $5\sqrt{2}$ cm

17



$A = xy$

$A = x\left(-\frac{3}{4}x + 6\right)$

$A = -\frac{3}{4}x^2 + 6x$

$A' = -\frac{3}{2}x + 6$

$0 = -\frac{3}{2}x + 6$

$-6 = -\frac{3}{2}x$

$-12 = -3x$

$4 = x$

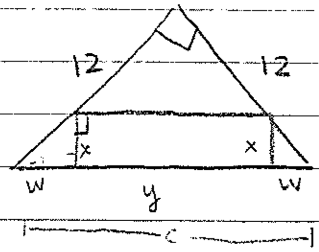
$y = -\frac{3}{4}(4) + 6$

$y = 3$

The dimensions are 4cm × 3cm

6.3 Continued

18.

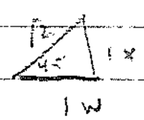


$c^2 = 12^2 + 12^2$

$c = \sqrt{288}$

$c = 12\sqrt{2}$

$W = x$



$2w + y = 12\sqrt{2}$

$2x + y = 12\sqrt{2}$

$y = 12\sqrt{2} - 2x$

The dimensions are $3\sqrt{2}$ cm × $6\sqrt{2}$ cm

$A = x \cdot y$

$A = x(12\sqrt{2} - 2x)$

$A = 12\sqrt{2}x - 2x^2$

$A' = 12\sqrt{2} - 4x$

$A' = 4(3\sqrt{2} - x)$

$0 = 4(3\sqrt{2} - x)$

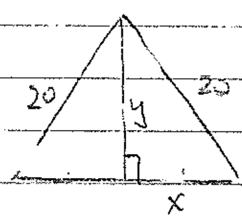
$x = 3\sqrt{2}$

$y = 12\sqrt{2} - 2(3\sqrt{2})$

$y = 12\sqrt{2} - 6\sqrt{2}$

$y = 6\sqrt{2}$

19.



$x^2 + y^2 = 20^2$

$y^2 = 400 - x^2$

$y = \sqrt{400 - x^2}$

$A = \frac{1}{2} bh$

$= \frac{1}{2}(2x)(y)$

$A = xy$

$A = x(\sqrt{400 - x^2})$

$A' = (1)(400 - x^2)^{1/2} + (x)(\frac{1}{2})(400 - x^2)^{-1/2}(-2x)$

$A' = (400 - x^2)^{-1/2}[400 - x^2 - x^2]$

$A' = \dfrac{400 - 2x^2}{(400 - x^2)^{1/2}}$

$0 = 2(200 - x^2)$

$0 = 2(10\sqrt{2} + x)(10\sqrt{2} - x)$

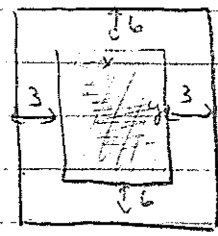
$x = -10\sqrt{2}$ $\quad x = 10\sqrt{2}$

$y = 10\sqrt{2}$

Max area $x \cdot y$

$(10\sqrt{2})(10\sqrt{2}) = 100 \cdot 2 = 200\text{ cm}^2$

6.3- Continued

20 

Inside area → $450cm^2$

$xy = 450$

$y = \frac{450}{x}$

Area $(12+y)(6+x)$

$s(x) = \left(12 + \frac{450}{x}\right)(6+x)$

$s'(x) = (-450x^{-2})(6+x) + (12+450x^{-1})(1)$

$s'(x) = \frac{-450(6+x)}{x^2} + 12 + \frac{450}{x}$

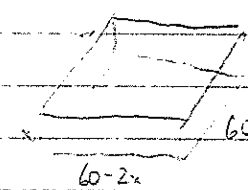
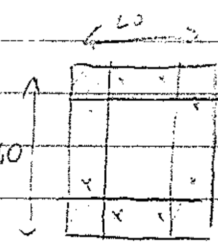
$s'(x) = \frac{-450(6+x) + 12x^2 + 450x}{x^2}$

$0 = -2700 - 450x + 12x^2 + 450x$

$0 = 12x^2 - 2700 = 12(x^2-225) = 12(x-15)(x+15)$

$x = 15$ $\quad y = \frac{450}{15}$ $\quad y =$

$x+6 = 21$ $\quad y+12 = 4$

21.  

$V = l \cdot w \cdot h$

$= (60-2x)(60-2x)(x)$

$= (60-2x)^2(x)$

$V' = 2(60-2x)(-2)(x) + (60-2x)^2(1)$

$V' = (60-2x)[-4x + 60 - 2x]$

$V' = (60-2x)(60-6x)$

$V' = 12(30-x)(10-x)$

$0 = 12(30-x)(10-x)$

$x = 30 \quad x = 10$

↑ reject $\quad 60-2x = 60-20 = 40$

too big

dimensions: $l \times w \times h$

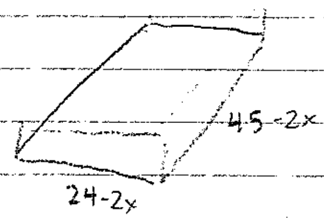
of box = $40 \times 40 \times 10$

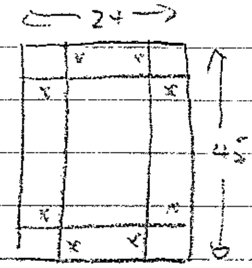
dimensions

of cutout → $10cm \times 10cm$

6.3 - Continued

22.





$V = l \cdot w \cdot h$

$V = (45-2x)(24-2x)(x)$

$V = (45-2x)(24x-2x^2)$

$V' = (-2)(24x-2x^2) + (45-2x)(24-4x)$

$V' = -48x + 4x^2 + 1080 - 180x - 48x + 8x^2$

$V' = 12x^2 - 276x + 1080$

$V' = x^2 - 23x + 90$

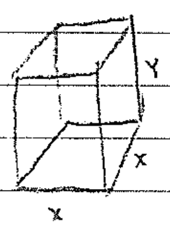
$V' = (x-18)(x-5)$

$x = 18$ ↑ $x > 12$, not (too big) $\quad x = 5$ $\quad 45-2(5) = 35$ $\quad 24-2(5) = 14$

dimensions of box: $5 \times 35 \times 14$

dimensions of cutout: 5cm x 5cm

23.



Volume → $32m^3$

$V = l \cdot w \cdot h$

$32 = x \cdot x \cdot y$

$y = \frac{32}{x^2}$

Surface Area

$A = 4xy + x^2$

$A = 4x\left(\frac{32}{x^2}\right) + x^2$

$A = \frac{128}{x} + x^2$

$A' = -128x^{-2} + 2x^1$

$0 = 2x^{-2}(x^3 - 64)$

$0 = 2x^{-2}(x-4)(x^2+4x+16)$

$x = 0,\ x = 4$

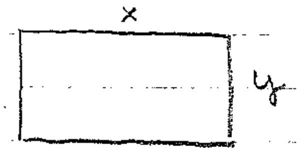
Solve for y: $y = \frac{32}{4^2}$

$y = 2$

$x = 4,\ y = 2$

6.3 - Continued

24.

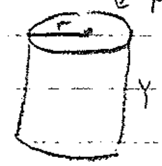


Perimeter = 216

$216 = 2x + 2y$

$108 = x + y$

$y = 108 - x$



perimeter of the top is x

(circumference) = x

$c = 2\pi r$

$x = 2\pi r$

$\frac{x}{2\pi} = r$

$$\text{Volume} = \pi r^2 h$$

$$= \pi \left(\frac{x}{2\pi}\right)^2 y$$

$$= \frac{x^2 y}{4\pi}$$

$$= \frac{x^2(108 - x)}{4\pi} = \frac{108x^2 - x^3}{4\pi}$$

$$V = \frac{27x^2}{\pi} - \frac{x^3}{4\pi}$$

$$V' = \frac{54x}{\pi} - \frac{3x^2}{4\pi} = 3x\left(18 - \tfrac{1}{4}x\right)$$

$x = 0 \qquad 18 - \frac{1}{4}x = 0$

$-\frac{1}{4}x = -18$

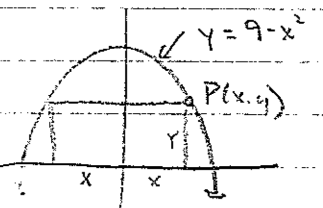
$x = 72$

$y = 108 - 72$

$y = 36$

72×36

25.



Area → $2xy$

$A = 2x(9 - x^2)$

$A = 18x - 2x^3$

$A' = 18 - 6x^2$

$A' = 6(3 - x^2)$

$0 = 6(\sqrt{3} + x)(\sqrt{3} - x)$

$x = -\sqrt{3} \qquad x = \sqrt{3}$

$y = 9 - (\sqrt{3})^2$

$y = 9 - 3$

$y = 6$

Area → $2xy$

$= 2(\sqrt{3})(6)$

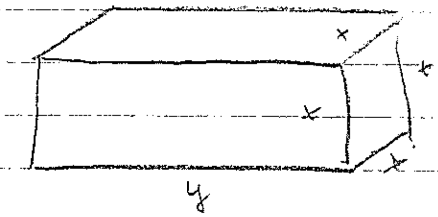
$= 12\sqrt{3}\ u^2$

6.3 - Continued

26 Girth + Length = $4x + y$

$300 = 4x + y$

$y = 300 - 4x$



$V = l \cdot w \cdot h$

$V = y \cdot x \cdot x$

$V = (300 - 4x)(x^2)$

$V' = (-4)(x^2) + (300 - 4x)(2x)$

$V' = -4x^2 + 600x - 8x^2$

$V' = -12x^2 + 600x$

$V' = -12x(x - 50)$

$x = 0 \quad x = 50$

$y = 300 - 4(50)$

$y = 100$

$V = (100)(50)(50)$

$V = 250{,}000 \text{ cm}^3$

27. Let x = price of ticket

\$20 = 3000 Fans $\quad A(20) = 3000 \quad$ A = Attendance

\$1 less = 500 fans $\quad A(P-1) = A(P) - 500 \quad$ P = price

Max rev at price P = $p \cdot$ Attendance

$= p \cdot A(p)$

linear function

$m = \dfrac{\Delta A}{\Delta p} = \dfrac{500}{-1} = -500$

$b = A(20) = 3000 = -500(20) + b \qquad Y = mx + b$

$b = 13{,}000$

linear function $A(p) = -500p + 13{,}000 \quad (y = mx + b)$

$R(p) = p \cdot A(p) = p(-500p + 13{,}000) = -500p^2 + 13000p$

$R'(p) = -1000p + 13000$

$0 = -1000(p - 13)$

$p = 13$

$A(13) = -500(13) + 13{,}000$

$= 6{,}500$

$R(13) = p \cdot A(p) = 13 \cdot A(13) = 13 \cdot 6500 = 84{,}500$