6.3 Optimization Problems

These problems often referred to as **max min** problems.

There are many real world applications of these types of problems.

Number Problems

Ex.1 Find two non-negative numbers whose sum is 15 such that the product of one with the square of the other is a maximum.

1 let x = 1st non negative #

y = 2nd " " "

$\boxed{x+y=15}$ 1 $\boxed{P = xy^2}$ *

$x = 15 - y$

$P = (15-y)(y^2)$

$P = 15y^2 - y^3$ 1

1 $P' = 30y - 3y^2$

$$30y - 3y^2 = 0$$

$$3y(10 - y) = 0$$

$y = 0$ or $y = 10$ (1)

Proof

$P'' = 30 - 6y$ (1)

$\cancel{P''(0) = 30 - 6(0) > 0}$

$P''(10) = 30 - 6(10) < 0$

$\therefore \cap \quad \therefore y = 10$ max

Find x

$x = 15 - y$

(1) $x = 15 - 10 = 5$

(1) The two #'s are 5, 10.

<u>Ex.2</u> **Two non-negative numbers have a product of 16. Find these numbers if the sum of their squares is to be a minimum. What is the minimum sum of the squares?**

let x = 1st non negative #

y = 2nd non negative #

$$\boxed{xy = 16} \qquad \boxed{S = x^2 + y^2} \; *$$

$$y = \frac{16}{x}$$

$$S = x^2 + \left(\frac{16}{x}\right)^2$$

$$S = x^2 + \frac{256}{x^2}$$

$$S = x^2 + 256x^{-2}$$

$$2x - 512x^{-3}$$

$$S' = 2x\frac{x^3}{x^3} - \frac{512}{x^3}$$

$$S' = \frac{2x^4 - 512}{x^3}$$

$$\underline{S' = 0}$$

$$2x^4 - 512 = 0$$

$$2x^4 = 512$$

$$x^4 = 256$$

$$x = \pm 4$$

$$x = 4$$

$$\underline{S' \infty}$$

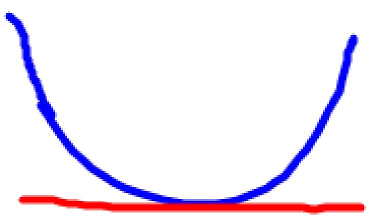
$$x^3 = 0$$

$$x = 0$$

Prob

$$S'' = 2 + \frac{1536}{x^4}$$

$$S''(4) = 2 + \frac{1536}{(4)^4} > 0$$



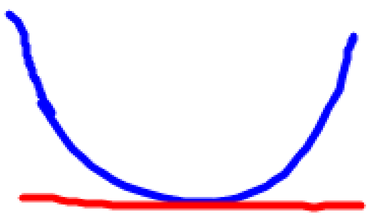
$$S''(0) \quad \infty$$

$$\boxed{x = 4 \text{ min}}$$

Prob

$$s'' = 2 + \frac{1536}{x^4}$$

$$s''(4) = 2 + \frac{1536}{(4)^4} > 0$$



$s''(0)$ ∞

$$\boxed{x = 4 \text{ min}}$$

Find

$$y = \frac{16}{x}$$

$$y = \frac{16}{4} = 4$$

$$S = x^2 + y^2$$

$$S = (4)^2 + (4)^2$$

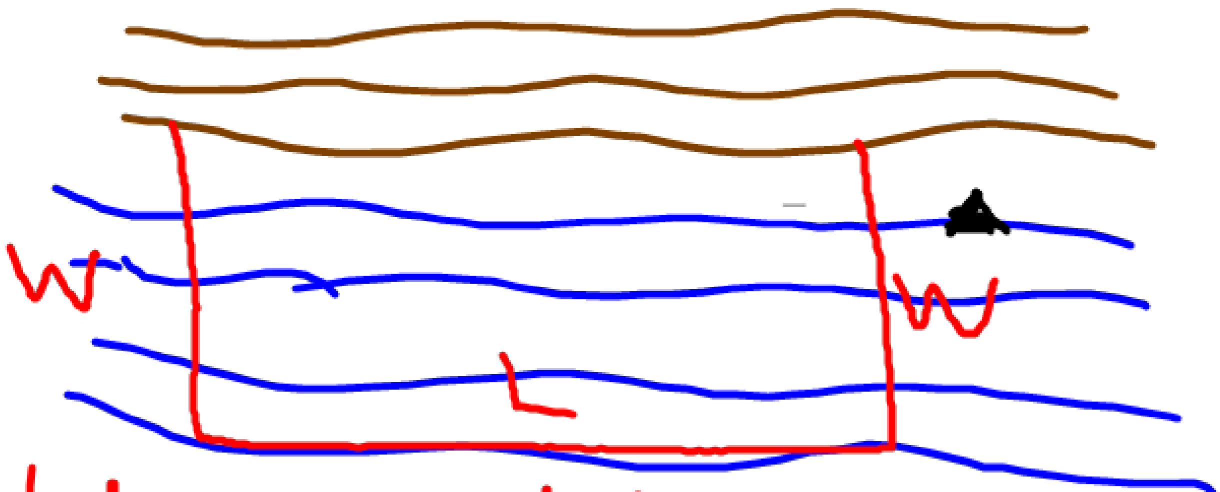
$$S = 32$$

The two non-negative #'s are 4 and 4 and minimum sum is 32.

Area Perimeter Problems

Example 3:

In a conservation park, a lifeguard has use 620 m of marker buoys to rope off a rectangular safe swim area. If one side of the area is the beach, calculate the dimensions of the swimming area so that it is a maximum.



let w = width

L = length

$$\boxed{2w + L = 620}$$

$$L = 620 - 2w$$

$$\boxed{A = WL} \; *$$

$A = w(620 - 2w)$

$A = 620w - 2w^2$

$A' = 620 - 4w$

$\underline{A' = 0}$

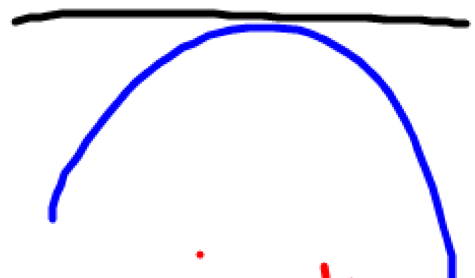
$620 - 4w = 0$

$620 = 4w$

$155 = w$

Proof

$A'' = -4$



$\therefore\ w = 155$ max

$L = 620 - 2w$

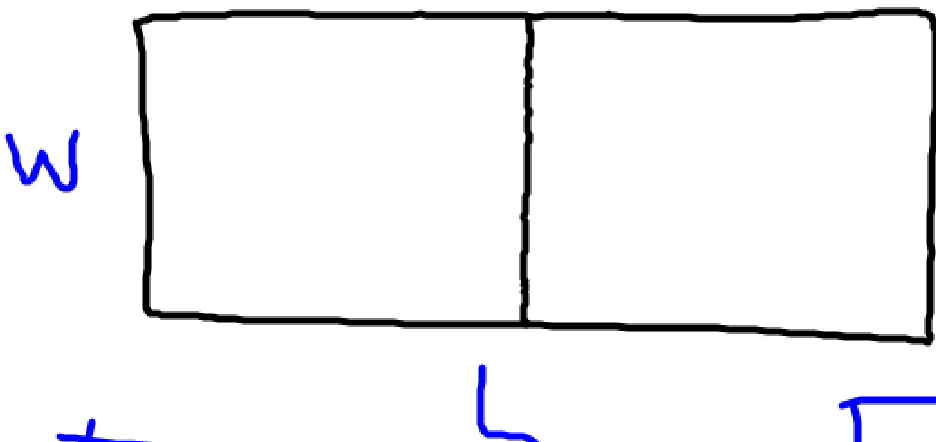
$L = 620 - 2(155)$

$L = 310$

Therefore the dimensions are $155\,m \times 310\,m$.

<u>Ex.4</u> A farmer has 1000m of fencing and wants to enclose a rectangular pasture bordering a river. Find the dimensions that will maximize the area. What is the maximum area?

Example 5 : **A farmer wants to build a 216 m^2 rectangular garden and divide it in half using another fence parallel to one side. What dimensions for the outer rectangle will require the smallest amount of fencing? How much fencing will be used?**



Let w = width

L = length

$$\boxed{216 = LW}$$

$$\boxed{P = 3W + 2L}$$

$$L = \frac{216}{W}$$

$$P = 3W + \frac{2}{1}\left(\frac{216}{W}\right)$$

$$P = 3w + 432w^{-1}$$

$$P' = 3 - 432w^{-2}$$

$$P' = 3 - \frac{432}{w^2}$$

$$\frac{P' \infty}{\cancel{w=}0}$$

$$\frac{P' = 0}{3 - \frac{432}{w^2} = 0}$$

$$3 = \frac{432}{w^2}$$

$$3w^2 = 432$$

$$w^2 = 144$$

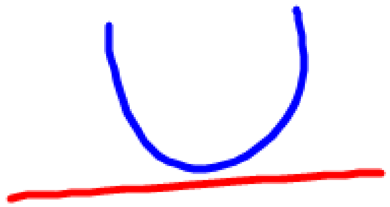
$$w = \pm 12$$

$$w = 12$$

Proof

$$P'' = \frac{864}{w^3}$$

$$P''(12) = \frac{864}{(12)^3} > 0$$



$\therefore w = 12$ min

Find L

$\frac{216}{w} = L$

$\frac{216}{12} = L$

$18 = L$

$P = 2L + 3W$

$P = 2(18) + 3(12)$

$P = 72m$

The dimensions are 18m by 12m and the minimum amount of fence is 72m.

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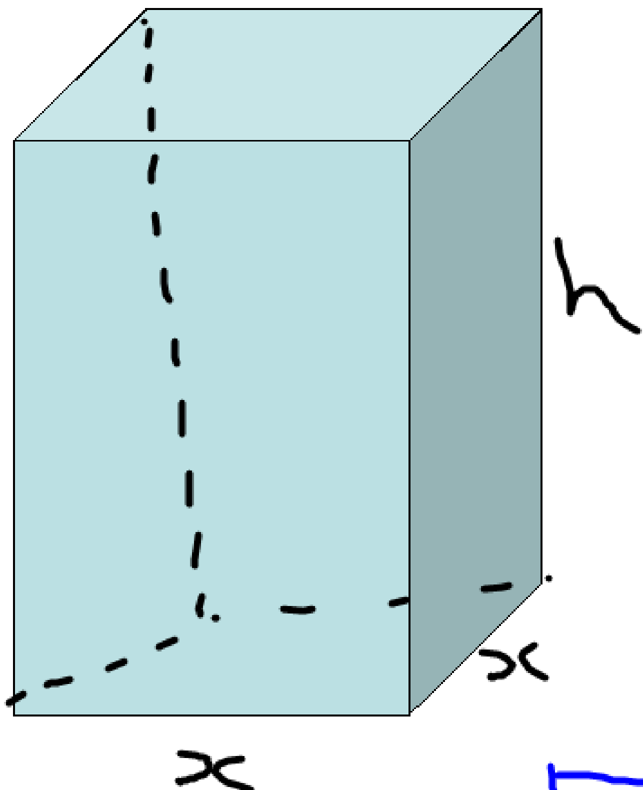
Volume Surface Area Problems

<u>Ex.4</u> If 200cm² of material is available to make a box with a square base and an open top. Find the largest possible volume of the box. What dimensions maximize the volume of the box?



let x = width / length

h = height

$$\boxed{V = x^2 h} \ *$$

SA

area base + area walls

$$\boxed{200 = x^2 + 4xh}$$

$$\frac{200-x^2}{4x} = \frac{4xh}{4x}$$

$$\boxed{\frac{200-x^2}{4x} = h}$$

$$V = x^{\cancel{2}}\left(\frac{200-x^2}{4\cancel{x}}\right)$$

$$V = \frac{200x-x^3}{4}$$

$$V = 50x - \frac{x^3}{4}$$

$$V' = 50 - \frac{3}{4}x^2$$

$$V' = 0$$

$$50 - \frac{3}{4}x^2 = 0$$

$$\left(50 = \frac{3}{4}x^2\right)4$$

$$200 = 3x^2$$

$$\frac{200}{3} = x^2$$

$$\pm\sqrt{\frac{200}{3}} = x$$

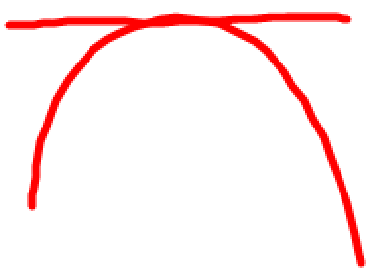
$$\pm 8.16 = x$$

$x = 8.16$

<u>Prob</u>

$$V'' = -\frac{6}{4}x$$

$$V''(8.16) = -\frac{6}{4}(8.16) < 0$$



max

$$h = \frac{200 - (8.16)^2}{4(8.16)}$$

$$h = 4.08 \text{ cm}$$

$$V = x^2 h$$

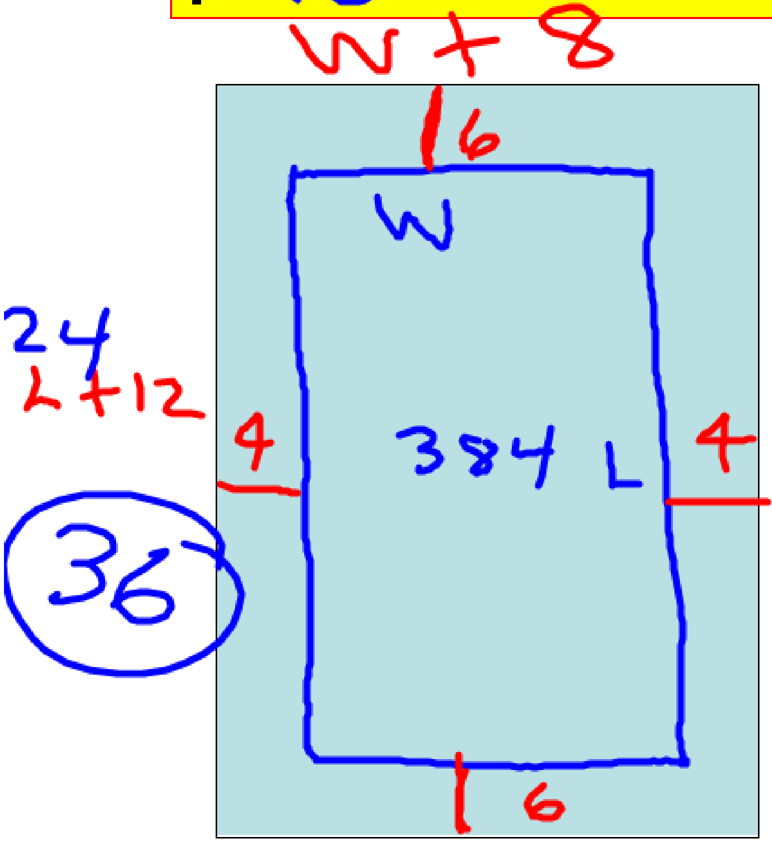
$$V = (8.16)^2(4.08)$$

$$V = 271.67 \text{ cm}^3$$

The dimensions of box are 8.16cm x 8.16cm x 4.08cm and volume is 271.67 cm3

Poster Problems

<u>Ex.5</u> **The top and bottom margins of a poster are six cm and the side margins are 4 cm. If the printed area on the poster is 384 cm^2, find the dimensions of the poster with the smallest area.**



Let w = width printed part

L = length of printed part.

$$\boxed{WL = 384}$$

$$A = (W+8)(L+12)$$

$$A = WL + 12W + 8L + 96$$

$$A = 384 + 12W + 8L + 96$$

$$\boxed{A = 480 + 12w + 8L} \quad *$$

$$L = \frac{384}{w}$$

$$A = 480 + 12w + 8\left(\frac{384}{w}\right)$$

$$A = 480 + 12w + 3072w^{-1}$$

$$A' = 12 - 3072w^{-2}$$

$$0 = 12 - \frac{3072}{w^2}$$

$$\frac{3072}{w^2} = 12$$

$$3072 = 12w^2$$

$$256 = w^2$$

$$\pm 16 = w$$

$$w = 16$$

<u>Proof</u>

$$A'' = \frac{6142}{w^3}$$

$$A''(16) = \frac{6142}{(16)^3} > 0$$

$$\cup$$

$\therefore$ $w = 16$ min

$$L = \frac{384}{w}$$

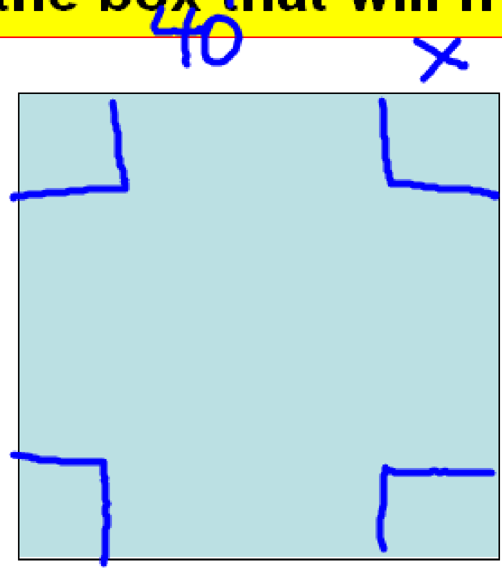
$$= \frac{384}{16}$$

$$L = 24$$

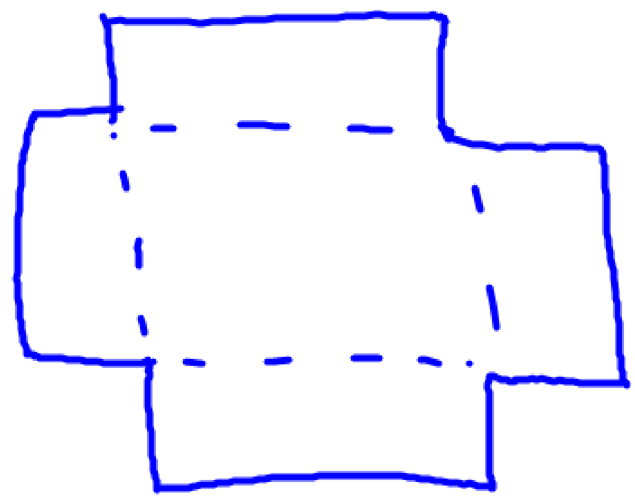
The dimensions of the poster are 24 cm x 36 cm.

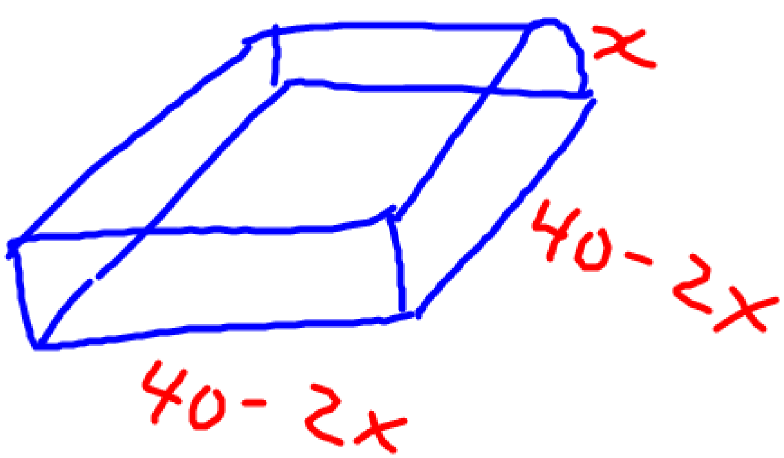
Volume Problems

<u>Ex.6</u> **A box is to be made from a square piece of cardboard by cutting a square out of each corner and turning the sides up to form walls. Given that the cardboard is 40cm by 40 cm find the dimensions of the box that will maximize the volume.**



let x = length cut





$$V = (40-2x)(40-2x)x$$

$$V = (1600 - 160x + 4x^2)x$$

$$V = 4x^3 - 160x^2 + 1600x$$

$$V' = 12x^2 - 320x + 1600$$

$$12x^2 - 320x + 1600 = x$$

quad formula

$\cancel{x = 20}$ or $x = 20/3$

$$V = (40-2x)(40-2x)x$$

$$V = (1600 - 160x + 4x^2)x$$

$$V = 4x^3 - 160x^2 + 1600x$$

$$V' = 12x^2 - 320x + 1600$$

$$12x^2 - 320x + 1600 = x$$

quad formula

$\cancel{x = 20}$ or $x = 20/3$

$$V = (40-2x)(40-2x)x$$

$$V = (1600 - 160x + 4x^2)x$$

$$V = 4x^3 - 160x^2 + 1600x$$

$$V' = 12x^2 - 320x + 1600$$

$$12x^2 - 320x + 1600 = x$$

quad formula

$\cancel{x = 20}$ or $x = 20/3$

$$V = (40-2x)(40-2x)x$$

$$V = (1600 - 160x + 4x^2)x$$

$$V = 4x^3 - 160x^2 + 1600x$$

$$V' = 12x^2 - 320x + 1600$$

$$12x^2 - 320x + 1600 = x$$

quad formula

$\cancel{x = 20}$ or $x = 20/3$

Proof

$$V'' = 24x - 320$$

$$V''\left(\frac{20}{3}\right) = 8\cancel{24}\left(\frac{20}{\cancel{3}}\right) - 320$$

$$= 160 - 320 < 0$$

$\therefore \cap$ CD

$\therefore x = \frac{20}{3}$ max.

Find W/L

$$W = L = 40 - 2x$$

$$= 40 - 2\left(\frac{20}{3}\right)$$

$$W = L = 26.67$$

The dimensions are 26.67 cm x 26.67 cm x 6.67 cm.

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