6.2 Sum, Difference and Double Angle Identities

A common misunderstanding of many students when studying identities is to think that:

$$\cos(40^\circ + 35^\circ) = \cos 40^\circ + \cos 35^\circ$$

Let's check it out with our calculator!

$$\cos(40^\circ + 35^\circ) = \cos 40^\circ + \cos 35^\circ$$

What does $cos(40° + 35°)$ equal?

$$\cos\left(40^\circ + 35^\circ\right) = \cos 40^\circ \cos 35^\circ - \sin 40^\circ \sin 35^\circ$$

In general $\cos(A + B) = cosAcosB - sinAsinB$

This is known as the **sum identity for cosine.**

Similarly it can be shown that:

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

This is known as the **difference identity for cosine.**

The sum and difference identities for sine are as follows:

$$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$$
$$\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$$

$$\sin(A \pm B) = \sin A\cos B \pm \sin B\cos A$$

The sum and difference identities for tangent are as follows:

$$\tan(\alpha \pm \beta) = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta}$$

Ex.1 Rewrite each of the following as the single trigonometric function of an angle:

$$a)\ \cos 110^\circ \cos 80^\circ - \sin 110^\circ \sin 80^\circ$$

$\cos(110^\circ + 80^\circ) = \cos 190^\circ$

$$b)\ \sin 75^\circ \cos 30^\circ + \cos 75^\circ \sin 30^\circ$$

$\sin(75^\circ + 30^\circ) = \sin 105^\circ$

$$c)\ \frac{\tan 5x + \tan 3x}{1 - \tan 5x \tan 3x}$$

$\tan(5x + 3x) = \tan 8x$

A special case occurs in the angle sum identities when A = B. Substituting B = A results in the double-angle identities.

Double Angle Identities

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$
$$= 1 - 2\sin^2\theta$$
$$= 2\cos^2\theta - 1$$
$$\sin 2\theta = 2\sin\theta\cos\theta$$
$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$$

Example 2: Write each expression as a **single** trigonometric function.

a) $2\sin 44° \cos 44°$

$= \sin 2(44°) = \sin 88°$

b) $\frac{2\tan 18°}{1-\tan^2 18°}$

$= \tan 2(18°) = \tan 36°$

c) $\cos^2 \frac{\pi}{5} - \sin^2 \frac{\pi}{5}$

$\cos 2\left(\frac{\pi}{5}\right) = \cos \frac{2\pi}{5}$

Your Turn

Write each expression as a single trigonometric function.

a) $\cos 88° \cos 35° + \sin 88° \sin 35°$

b) $2 \sin \frac{\pi}{12} \cos \frac{\pi}{12}$

a) $\cos(88° - 35°) = \cos 53°$

b) $\sin 2(\pi/12) = \sin \pi/6$

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③ $\underbrace{1 - \cos 2x}$

$= 1 - (1 - 2\sin^2 x)$

$= 1 - 1 + 2\sin^2 x$

$= \underline{2\sin^2 x}$

4b)

$$\left(6\cos^2 24^\circ - 6\sin^2 24^\circ\right)\tan 48^\circ$$

$$= 6\left[\cos^2 24^\circ - \sin^2 24^\circ\right]\tan 48^\circ$$

$$= 6\left[\cos 2(24^\circ)\right]\tan 48^\circ$$

$$= 6(\cos 48^\circ)(\tan 48^\circ)$$

$$6\cancel{\cos 48^\circ}\left(\frac{\sin 48^\circ}{\cancel{\cos 48^\circ}}\right)$$

$$= 6\sin 48^\circ$$

4e) $1 - 2\cos^2 \pi/12$

$$= -\left(-1 + 2\cos^2 \pi/12\right)$$

$$= -\left(2\cos^2 \pi/12 - 1\right) = -\cos \pi/6$$

Example 3: Consider the expression $\frac{1-\cos 2x}{\sin 2x}$

a) What are the non-permissible values for the expression?

b) Simplify the expression to one of the three primary trig functions.

a) $\sin 2x$

$= 2\underline{\sin x}\,\underline{\cos x}$

<u>NPV</u>

$\sin x \neq 0$

$0 + \pi n, n \in I$

$\cos x \neq 0$

$\frac{\pi}{2} + \pi n, n \in I$

b) $\dfrac{1-\cos 2x}{\sin 2x}$

$\sin x$
$\cos x$
$\tan x$

$= \dfrac{1-(1-2\sin^2 x)}{2\sin x\cos x}$

$= \dfrac{1-1+2\sin^2 x}{2\sin x\cos x} = \dfrac{\cancel{2}\sin^{\cancel{2}} x}{\cancel{2}\cancel{\sin x}\cos x} = \tan x$

Your turn: Simplify the following to one of the primary trigonometric ratios.

$$\frac{sin2x}{cos2x + 1}$$

$$\frac{2\sin x\cos x}{2\cos^2 x - 1 + 1}$$

$$\frac{\cancel{2}\sin x\,\cancel{\cos x}}{\cancel{2}\cos^{\cancel{2}} x} = \tan x$$

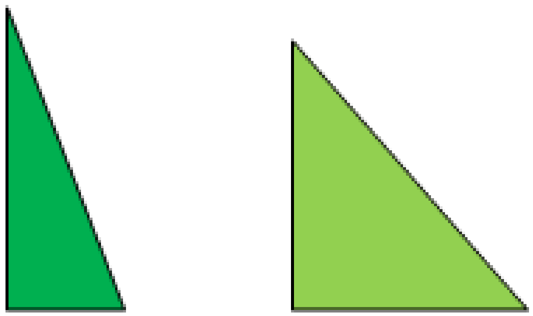
$$\frac{2\sin x\cos x}{(1-2\sin^2 x)+1}$$

$$\frac{2\sin x\cos x}{2-2\sin^2 x}$$

$$= \frac{2\sin x\cos x}{2(1-\sin^2 x)} = \frac{\cancel{2}\sin \cancel{\cos x}}{\cancel{2}\cos^{\cancel{2}} x}$$

Ex. 4 Find the $\boxed{\text{exact value}}$ of the sin105°.

$$\sin(60^\circ + 45^\circ)$$



Recall our reference triangles.

$$= (\sin 60^\circ)(\cos 45^\circ) + (\sin 45^\circ)(\cos 60^\circ)$$

$$= \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) + \left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right)$$

$$= \frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4} = \frac{\sqrt{6}+\sqrt{2}}{4}$$

Your Turn

Use a sum or difference identity to find the exact values of

a) $\cos 165°$ **b)** $\tan \frac{11\pi}{12}$ **c)** $tan 15°$

a) $\cos(120° + 45°)$

$= (\cos 120)(\cos 45) - (\sin 120)(\sin 45)$

$= \left(-\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right) - \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right)$

$= -\frac{\sqrt{2}}{4} - \frac{\sqrt{6}}{4} = \frac{-\sqrt{2}-\sqrt{6}}{4}$

$$\frac{11\cancel{\pi}}{12} \cdot \frac{\cancel{180}^{\circ}}{\cancel{\pi}} = 165^{\circ}$$

$$\tan 165^{\circ}$$

$$\tan(120^{\circ} + 45^{\circ}) = \frac{\tan 120^{\circ} + \tan 45^{\circ}}{1 - (\tan 120^{\circ})(\tan 45^{\circ})}$$

$$= \frac{-\sqrt{3} + 1}{1 - (-\sqrt{3})(1)}$$

$$= \frac{(1-\sqrt{3})}{(1+\sqrt{3})} \frac{(1-\sqrt{3})}{(1-\sqrt{3})}$$

$$= \frac{1-\sqrt{3}-\sqrt{3}+3}{1-3}$$

$$= \frac{4-2\sqrt{3}}{-2} = -2+\sqrt{3}$$

Example 5 :

α is a fourth quadrant angle and $\tan\alpha = \dfrac{-12}{5}$.

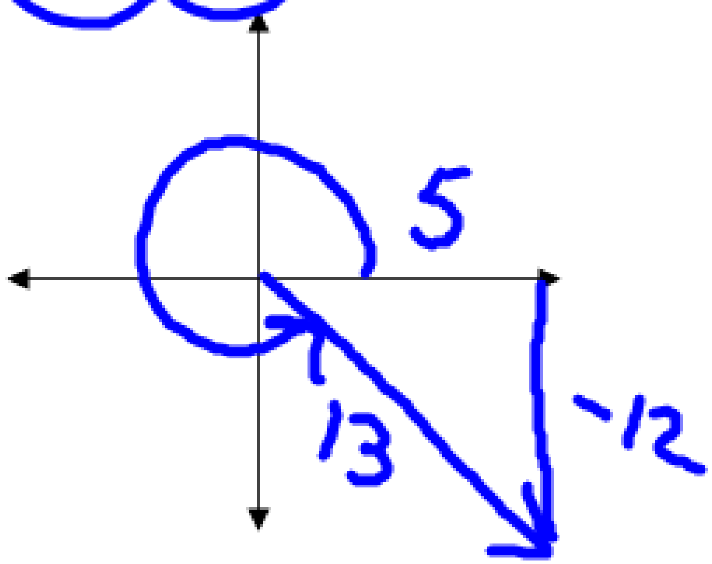
β is a second quadrant angle and $\sin\beta = \dfrac{3}{5}$.

a) Find the exact value of $\sin(\alpha + \beta)$

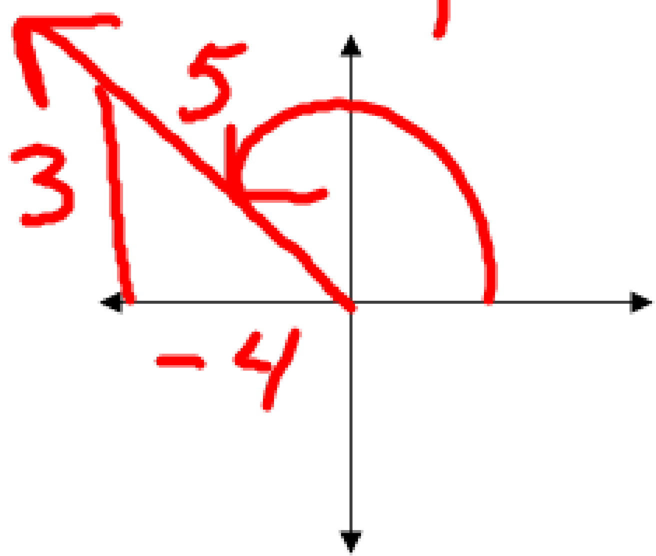
b) Find the exact value of $\cos(\alpha - \beta)$

c) Find the exact value of $\cos 2\alpha$

α —



β



a) $\sin(\alpha+\beta) = (\sin\alpha)(\cos\beta) + (\sin\beta)(\cos\alpha)$

$$= \left(-\frac{12}{13}\right)\left(-\frac{4}{5}\right) + \left(\frac{3}{5}\right)\left(\frac{5}{13}\right)$$

$$= \frac{48}{65} + \frac{15}{65} = \frac{63}{65}$$

b) $\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

$$= \left(\frac{5}{13}\right)\left(-\frac{4}{5}\right) + \left(-\frac{12}{13}\right)\left(\frac{3}{5}\right)$$

$$= -\frac{20}{65} - \frac{36}{65} = -\frac{56}{65}$$

c)

$$\cos 2\alpha = 2\cos^2\alpha - 1$$

$$= 2\left(\frac{5}{13}\right)^2 - 1$$

$$= 2\left(\frac{25}{169}\right) - 1$$

$$= \frac{50}{169} - \frac{169}{169}$$

$$= -\frac{119}{169}$$

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