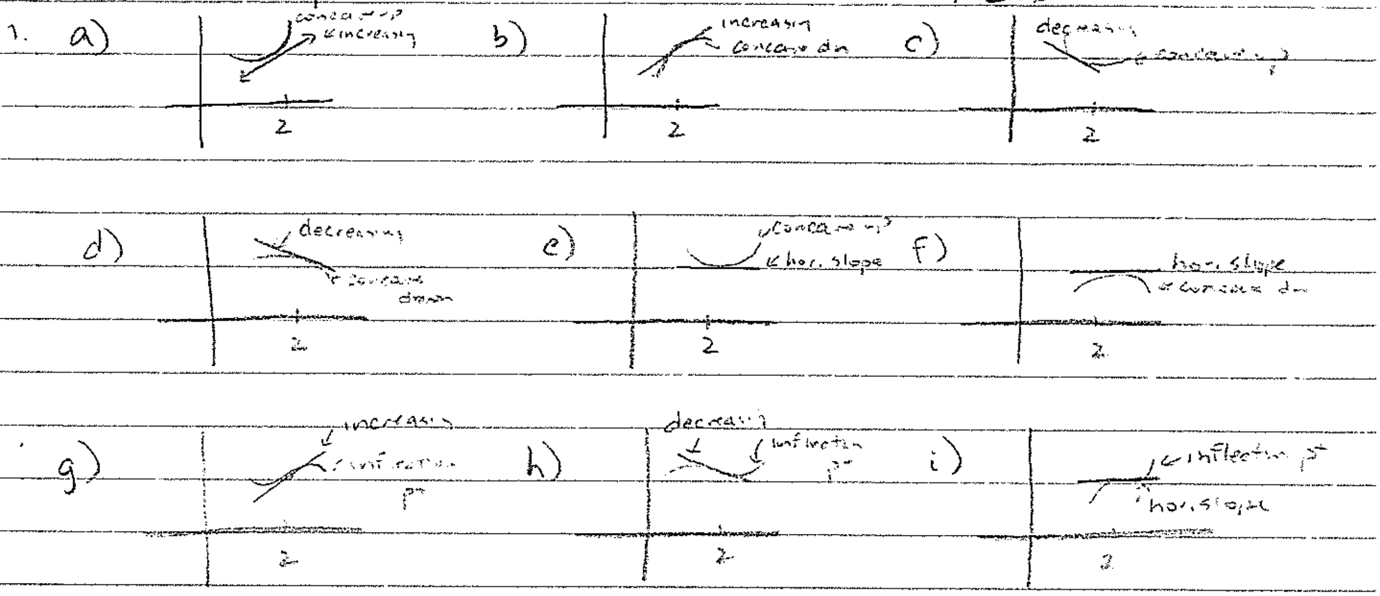
5.4 Concavity and the Second Derivative Test. P. 246 1-19

1.



2. a) $x \in (-2,-1) \cup (1,3)$ since $f''(x) > 0$ (concave up)

b) $x \in (-1,1)$ because $f''(x) < 0$ (concave down)

c) $x = \pm 1$ because $f''(x)$ switches sign

3. a) $x \in (-3,1)$ since $f'(x) > 0$ and $f'(x) = 0$ only at $x = -2$.

b) $x \in (1,3)$ since $f'(x) < 0$

c) relative max at $x = 1$

d) $x \in (-2,0) \cup (2,3)$ since $f'(x)$ is increasing

e) $x \in (-3,-2) \cup (0,2)$ since $f'(x)$ is decreasing

f) $x = -2$ and $x = 2$ since $f'(x)$ switches from increasing to decreasing

4. $f(x) = x^2 - 5x - 14$

$f'(x) = 2x - 5$

$f''(x) = 2$ ← concave up $(-\infty, \infty)$

no inflection pts

5. $f(x) = \frac{1}{3}x^3 + 2x^2$

$f'(x) = x^2 + 4x$

$f''(x) = 2x + 4$

$f''(x) = 2(x+2)$

$- - - - \;|\; + + + +$ $(x+2)$

$- - - - \; -2 \; + + + +$

concave up $(-2, \infty)$

concave down $(-\infty, -2)$

inflection pt $\left(-2, \frac{16}{3}\right)$

5.4 continued

6. $f(x) = \frac{1}{4}x^4 - 6x^2$

$f'(x) = x^3 - 12x$

$f''(x) = 3x^2 - 12$

$f''(x) = 3(x^2-4)$

$f''(x) = 3(x+2)(x-2)$

- - - 0 + + + + + $(x+2)$

- - - - 0 + + $(x-2)$

+ + -2 - - 2 + +

Concave up $(-\infty, -2) \cup (2, \infty)$

Concave down $(-2, 2)$

Inflection pt's $(2, -20)$

$(-2, -20)$

7. $f(x) = \frac{1}{12}x^4 + \frac{1}{6}x^3 - 3x^2 - 6x + 4$

$f'(x) = \frac{1}{3}x^3 + \frac{1}{2}x^2 - 6x - 6$

$f''(x) = x^2 + x - 6$

$f''(x) = (x+3)(x-2)$

- - - 0 + + + + + $(x+3)$

- - - - - - 0 + + + $(x-2)$

+ + + -3 - - 2 + + +

concave up $(-\infty, -3) \cup (2, \infty)$

concave down $(-3, 2)$

inflection points $(-3, -11/4)$

$(2, -52/3)$

8. $f(x) = 2x^3 - x^4$

$f'(x) = 6x^2 - 4x^3$

$f''(x) = 12x - 12x^2$

$f''(x) = 12x(1-x)$

- - - 0 + + + + + x

+ + + + + 0 - - - $1-x$

- - - 0 + + 1 - - -

Concave up $(0, 1)$

Concave down $(-\infty, 0) \cup (1, \infty)$

Inflection pts $(0, 0)$

$(1, 1)$

9. $f(x) = 3x^5 - 10x^3$

$f'(x) = 15x^4 - 30x^2$

$f''(x) = 60x^3 - 60x$

$f''(x) = 60x(x^2-1)$

$f''(x) = 60x(x+1)(x-1)$

- - - - 0 + + + + x

- - - 0 + + + + + $x+1$

- - - - - - - 0 + + $x-1$

- - - -1 + + 0 - - 1 + +

Concave up $(-1, 0) \cup (1, \infty)$

concave down $(-\infty, -1) \cup (0, 1)$

Inflection pts $(-1, 7)$

$(0, 0)$

$(1, -7)$

5-4 continued

10. $f(x) = x - 12\sqrt[3]{x}$

$f(x) = x - 12x^{1/3}$ — Concave up $(0, \infty)$

$f'(x) = 1 - 4x^{-2/3}$ — Concave down $(-\infty, 0)$

$f''(x) = \frac{8}{3}x^{-5/3}$ — Inflection pt $(0,0)$

$- - - \; 0 \; + + + \quad \frac{8}{3}x^{-5/3}$

$- - - \quad | \quad + + +$

11. $f(x) = \frac{1}{3}x^3 + 9x^{2/3}$

$f'(x) = x^2 + 6x^{-1/3}$ — Concave up $(1, \infty)$

$f''(x) = 2x - 2x^{-4/3}$ — Concave down $(-\infty, 0) \cup (0, 1)$

$f''(x) = 2x^{-4/3}(x^{7/3} - 1)$ — Inflection point $(1, \frac{28}{3})$

$+ + + + 0 + + + + \quad 2x^{-4/3} \quad \frac{2}{\sqrt[3]{x}}$

$- - - - - - 0 + + \quad x^{7/3} - 1$

$- - - - - 0 - - 1 + +$

12. $f(x) = \dfrac{9}{x^2+3} = 9(x^2+3)^{-1}$

$f'(x) = -9(x^2+3)^{-2}(2x)$

$f'(x) = -18x(x^2+3)^{-2} = \dfrac{-18x}{(x^2+3)^2}$

$f''(x) = \dfrac{-18(x^2+3)^2 - (-18x)(2)(x^2+3)(2x)}{(x^2+3)^4}$

$= \dfrac{-18[x^2+3-4x^2]}{(x^2+3)^3}$

$= \dfrac{-18[-3x^2+3]}{(x^2+3)^3} = \dfrac{54(x^2-1)}{(x^2+3)^3} = \dfrac{54(x+1)(x-1)}{(x^2+3)^3}$

$- - \; 0 + + + + \quad (x+1)$

$- - - - 0 + + \quad (x-1)$

$+ + + + + + + \quad (x^2+3)^3$

$+ + + \; -1 \; - - - \; 1 \; + + +$

Concave up $(-\infty, -1) \cup (1, \infty)$

Concave dn $(-1, 1)$

inflection pt $(1, 9/4)$ $(-1, \frac{9}{4})$

13. $f(x) = x - 1 + \dfrac{2}{x} = x - 1 + 2x^{-1}$

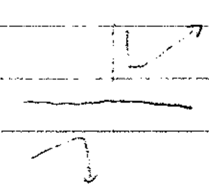
$f'(x) = 1 - 2x^{-2}$ — Concave up $(0, \infty)$

$f''(x) = 4x^{-3}$ — Concave down $(-\infty, 0)$

$- - - \; | \; + + + \quad 4x^{-3} = \dfrac{4}{\sqrt[3]{x}}$ — no inflection point

$- - - \; \underset{\text{und}}{0} \; + + +$ — because undefined at $x=0$

hts backwards



← on graphing calculator

5.4 Continued

14. $f(x) = \sqrt{x}\,(6+x)$

$f'(x) = \frac{1}{2}x^{-1/2}(6+x) + x^{1/2}(1)$

$f'(x) = 3x^{-1/2} + \frac{1}{2}x^{1/2} + x^{1/2}$

$f'(x) = 3x^{-1/2} + \frac{3}{2}x^{1/2}$

$f'(x) = \frac{3}{2}x^{-1/2}(2+x)$

$f''(x) = -\frac{3}{4}x^{-3/2}(2+x) + \frac{3}{2}x^{-1/2}(1)$

$f''(x) = -\frac{3}{4}x^{-3/2}\left[(2+x) - 2x\right]$

$f''(x) = -\frac{3}{4}x^{-3/2}(2-x)$

Concave up $(2, \infty)$

Concave down $(0, 2)$

inflection pt

$(2, 8\sqrt{2})$

+ + + + + + + 0 - - - - $(2-x)$

und 0 - - - - - - - - $-\frac{3}{4}x^{-3/2}$

0 2

und - - - - + + + + + +

15. $f(x) = \dfrac{x}{\sqrt{x-3}}$

$f'(x) = \dfrac{1(x-3)^{1/2} - (x)(\frac{1}{2})(x-3)^{-1/2}(1)}{(x-3)}$

$f'(x) = \dfrac{\frac{1}{2}(x-3)^{-1/2}\left(2(x-3) - x\right)}{(x-3)}$

$f'(x) = \frac{1}{2}(x-3)^{-3/2}(2x-6-x)$

$f'(x) = \frac{1}{2}(x-3)^{-3/2}(x-6) = \frac{1}{2}(x-6)(x-3)^{-3/2}$

$f''(x) = (\frac{1}{2})(x-3)^{-3/2} + (\frac{1}{2})(x-6)(-\frac{3}{2})(x-3)^{-5/2}(1)$

$f''(x) = -\frac{1}{4}(x-3)^{-5/2}\left[-2(x-3) + 3(x-6)\right]$

$f''(x) = -\frac{1}{4}(x-3)^{-5/2}\left[-2x + 6 + 3x - 18\right]$

$f''(x) = -\frac{1}{4}(x-3)^{-5/2}(x-12)$

Concave up $(3, 12)$

Concave down $(12, \infty)$

- - - - - - - - $-\frac{1}{4}$

und 0 + + + + + $(x-3)^{-5/2}$

- - - - - 0 + + + $(x-12)$

und 3 + + + 12 - - -

16. $f(x) = 2x^3 - 15x^2 + 9$

$f'(x) = 6x^2 - 30x$

$f'(x) = 6x(x-5)$ $f'(x) = 0$ at $\{0, 5\}$

- - - - 0 + + + + $6x$

- - - - - - 0 + + + $x-5$

+ + + 0 - - - 5 + + +

$f''(x) = 12x - 30$

$f''(x) = 6(2x-5)$

- - - - 0 + + + + $6(2x-5)$

- - - $\frac{5}{2}$ + + + +

$f''(0) = 12(0) - 30$

$= -30$ so $x = 0$

concave down

local maximum

$f''(5) = 12(5) - 30$

$= 30$ so $x = 5$

concave up

local minimum

5.4 - Continued

17. $f(x) = \frac{1}{4}x^4 - x^3 - \frac{1}{2}x^2 + 3x$

$f'(x) = x^3 - 3x^2 - x + 3$ critical #'s

$f'(x) = (x+1)(x-1)(x-3)$ $x=1,\ x=-1,\ x=3$

$f''(x) = 3x^2 - 6x - 1$

$f''(1) = 3(1)^2 - 6(1) - 1$ $f''(-1) = 3(-1)^2 - 6(-1) - 1$ $f''(3) = 3(3)^2 - 6(3) -$

$= -4$ (concave down) $= 8$ (concave up) $= 8$ (concave up)

∩ ↑ relative max ∪ ↑ relative min ∪ ↑ rel. min

18. $f(x) = x^5 - 5x^3$

$f'(x) = 5x^4 - 15x^2$

$f'(x) = 5x^2(x^2 - 3)$

$f'(x) = 5x^2(x+\sqrt{3})(x-\sqrt{3})$

$f''(x) = 20x^3 - 30x$

$f''(0) = 20(0)^3 - 30(0)$ $f''(\sqrt{3}) = 20(\sqrt{3})^3 - 30(\sqrt{3})$ $f''(-\sqrt{3}) = 20(-\sqrt{3})^3 - 30(-\sqrt{3})$

$= 0$ (inflection point) $= 60\sqrt{3} - 30\sqrt{3}$ $= -60\sqrt{3} + 30\sqrt{3}$

second derivative test fails $= 30\sqrt{3}$ ← rel. min $= -30\sqrt{3}$ ↑

∪ concave up ∩ concave down rel max

19. $f(x) = \dfrac{x^3}{x+2}$

$f'(x) = \dfrac{3x^2(x+2) - x^3(1)}{(x+2)^2}$ $f''(x) = 0$ for $\{-3, 0\}$

$f'(x) = \dfrac{3x^3 + 6x^2 - x^3}{(x+2)^2}$

$f'(x) = \dfrac{2x^3 + 6x^2}{(x+2)^2}$

$f'(x) = \dfrac{2x^2(x+3)}{(x+2)^2} = \dfrac{2x^3 + 6x^2}{(x+2)^2}$

$f''(x) = \dfrac{(6x^2 + 12x)(x+2)^2 - (2x^3 + 6x^2)(2)(x+2)}{(x+2)^4}$

$f''(x) = \dfrac{6x(x+2)(x+2)^2 - 4x^2(x+3)(x+2)}{(x+2)^4}$

$f''(x) = \dfrac{6x(x+2)^2 - 4x^3(x+2)}{(x+2)^3}$

$f''(3) = \dfrac{6(3)(3+2)^2 - 4(3)^3(3+2)}{(3+2)^3} = 18$ $f''(0) = 0$

∪ ↑ rel min ↑ second derivative test fails