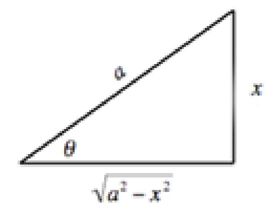
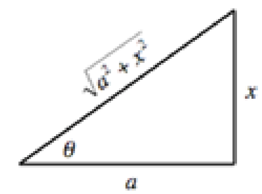
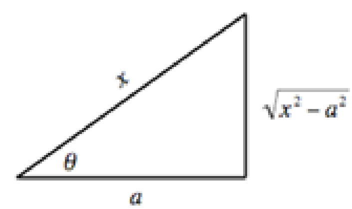
5.4 Integration By Trigonometric Substitution

Integration By Trigonometric Substitution

If the question is in the form of $\sqrt{a^2 - x^2}$, $\sqrt{x^2 - a^2}$, $\sqrt{x^2 + a^2}$ then we use trigonometric functions and identities to eliminate the square root.

Table of Substitutions

Expression	Substitution	Identity	Diagram
$\sqrt{a^2 - x^2}$	$x = asin\theta$	$1 - sin^2\theta = cos^2\theta$	
$\sqrt{a^2 + x^2}$	$x = atan\theta$	$1 + tan^2\theta = sec^2\theta$	
$\sqrt{x^2 - a^2}$	$x = asec\theta$	$sec^2\theta - 1 = tan^2\theta$	

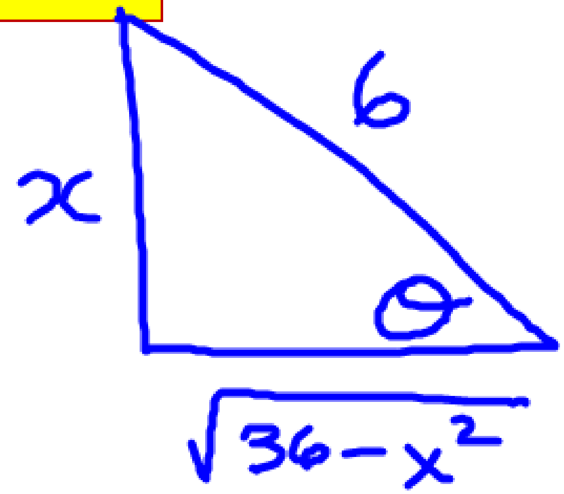
Example: $\int \frac{1}{\sqrt{36-x^2}}\,dx$

$$\boxed{x = 6\sin\theta}$$

$$dx = 6\cos\theta\, d\theta$$

$$\int \frac{\cancel{6\cos\theta}\, d\theta}{\cancel{6\cos\theta}}$$

$$\int 1\, d\theta = \theta + c$$



$$6\cos\theta = \sqrt{36-x^2}$$

$$= \theta + C$$

$$= \sin^{-1}\left(\frac{x}{6}\right) + C$$

$$x = 6\sin\theta$$

$$\frac{x}{6} = \sin\theta$$

$$\sin^{-1}\left(\frac{x}{6}\right) = \theta$$

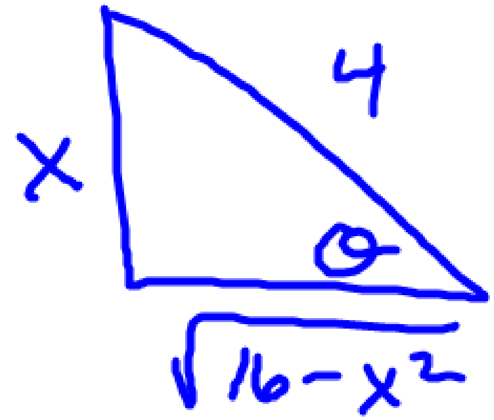
Example: $\int \frac{1}{x^2\sqrt{16-x^2}}\,dx$

$x = 4\sin\theta$

$dx = 4\cos\theta\,d\theta$

$$\int \frac{\cancel{4\cos\theta}\,d\theta}{(4\sin\theta)^2\,\cancel{4\cos\theta}}$$

$$\int \frac{1}{16\sin^2\theta}\,d\theta = \frac{1}{16}\int \csc^2\theta\,d\theta$$



$4\cos\theta = \sqrt{16-x^2}$

$$\frac{1}{16}\int \csc^2\theta \, d\theta$$

$$= \frac{1}{16}\left[-\cot\theta\right] + C$$

$$= -\frac{1}{16}\left[\frac{\sqrt{16-x^2}}{x}\right] + C$$

Example: $\int \frac{1}{\sqrt{4x^2+9}}\,dx$

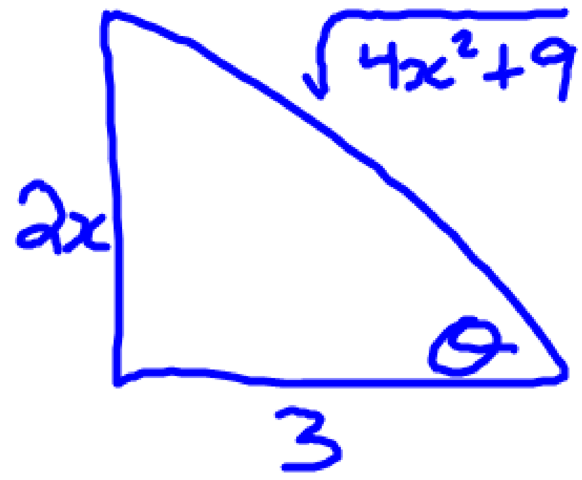
$\sqrt{a^2+x^2}$

$\sqrt{(3)^2+(2x)^2}$

$2x = 3\tan\theta$

$x = \frac{3}{2}\tan\theta$

$dx = \frac{3}{2}\sec^2\theta\, d\theta$



$\int \frac{\frac{3}{2}\sec^2\theta\, d\theta}{3\sec\theta}$

$\sec\theta = \frac{\sqrt{4x^2+9}}{3}$

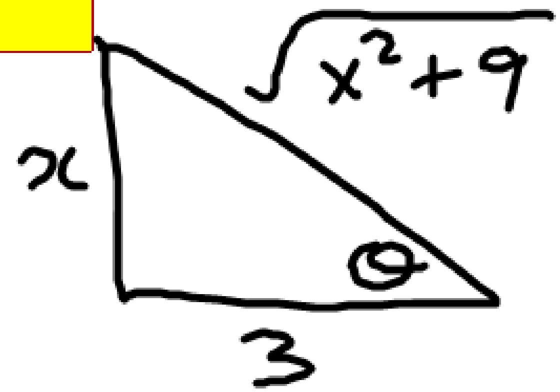
$3\sec\theta = \sqrt{4x^2+9}$

$$\frac{1}{2}\int \sec\theta \, d\theta$$

$$= \frac{1}{2}\Big[\ln\left|\sec\theta + \tan\theta\right|\Big] + C$$

$$= \frac{1}{2}\left[\ln\left|\frac{\sqrt{2x^2+9}}{3} + \frac{2x}{3}\right|\right] + C$$

Example: $\int \frac{4}{x^2\sqrt{x^2+9}}\,dx$



$$x = 3\tan\theta$$

$$dx = 3\sec^2\theta\, d\theta$$

$$4\int \frac{\cancel{3}\sec^{\cancel{2}}\theta\, d\theta}{(3\tan\theta)^2\ \cancel{3\sec\theta}}$$

$$\frac{4}{9}\int \frac{\sec\theta}{\tan^2\theta}\,d\theta$$

$$\sec\theta = \frac{\sqrt{x^2+9}}{3}$$

$$3\sec\theta = \sqrt{x^2+9}$$

$$= \frac{4}{9} \int \left(\frac{\cos^{\cancel{2}} \theta}{\sin^2 \theta} \right) \frac{1}{\cancel{\cos \theta}} \, d\theta$$

$$= \frac{4}{9} \int \frac{\cos \theta}{\sin^2 \theta} \, d\theta = \frac{4}{9} \int \frac{du}{u^2}$$

let $u = \sin \theta$

$du = \cos \theta \, d\theta$

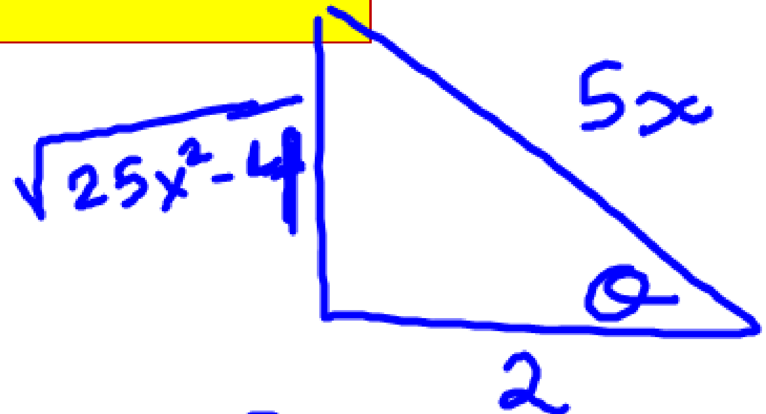
$$= \frac{4}{9} \left(-u^{-1} \right) + C$$

$$= \frac{-4}{9u} + C$$

$$= \frac{-4}{9(\sin\theta)} + C$$

$$= \frac{-4}{9\left(\frac{x}{\sqrt{x^2+9}}\right)} + C = \frac{-4\sqrt{x^2+9}}{9x} + C$$

Example: $\int \frac{\sqrt{25x^2-4}}{x}\,dx$



$$5x = 2\sec\theta$$

$$x = \frac{2}{5}\sec\theta$$

$$dx = \frac{2}{5}\sec\theta\tan\theta$$

$$2\tan\theta = \sqrt{25x^2-4}$$

$$\int \frac{2\tan\theta\left(\cancel{\tfrac{2}{5}}\,\cancel{\sec\theta}\tan\theta\right)d\theta}{\cancel{\tfrac{2}{5}}\,\cancel{\sec\theta}}$$

$$2\int \tan^2\theta \, d\theta$$

$$2\int (\sec^2\theta - 1)\, d\theta$$

$$= 2\left[\tan\theta - \theta\right] + C$$

$$= 2\left[\frac{\sqrt{25x^2-4}}{2} - \sec^{-1}\left(\frac{5x}{2}\right)\right] + C$$

Example: $\int \frac{x}{\sqrt{3-2x-x^2}}\,dx$

ASSIGNMENT HANDOUT