Theorem 4 (continued) The Fundamental Theorem of Calculus, Part 2

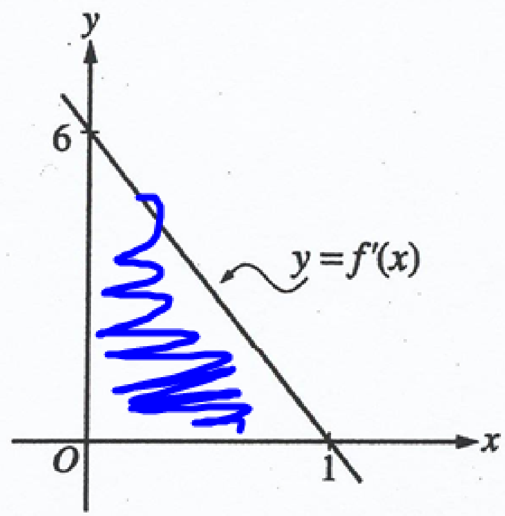
If f is continuous at every point of $[a, b]$, and if F is any antiderivative of f on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

This part of the Fundamental Theorem is also called the **Integral Evaluation Theorem.**

We did this in Section 5.3 when we evaluated integrals manually!

2003 AP MC Question



22. The graph of f', the derivative of f, is the line shown in the figure above. If $f(0) = 5$, then $f(1) =$

(A) 0 (B) 3 (C) 6 (D) 8 (E) 11

$\int_0^1 f' = f(1) - f(0)$

$\frac{1}{2}(1)(6) = f(1) - 5$

$3 = f(1) - 5$

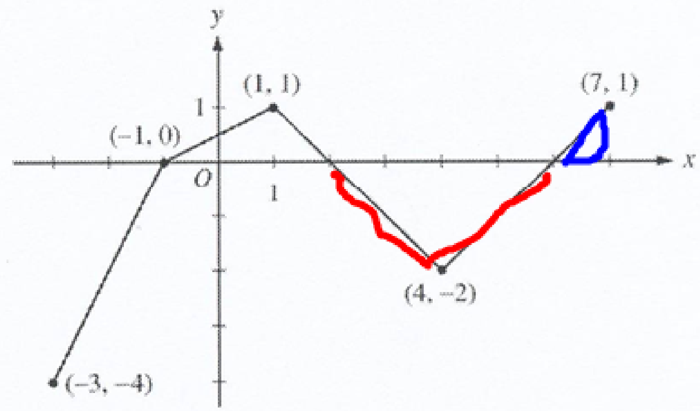
$f(1) = 8$

g' not diff [-3, 7] ∴ MVT does not apply.

AP® CALCULUS AB
2008 SCORING GUIDELINES (Form B)

Question 5

Let g be a continuous function with $g(2) = 5$. The graph of the piecewise-linear function g', the derivative of g, is shown above for $-3 \le x \le 7$.



Graph of g'

(a) Find the x-coordinate of all points of inflection of the graph of $y = g(x)$ for $-3 < x < 7$. Justify your answer.

(b) Find the absolute maximum value of g on the interval $-3 \le x \le 7$. Justify your answer.

(c) Find the average rate of change of $g(x)$ on the interval $-3 \le x \le 7$.

(d) Find the average rate of change of $g'(x)$ on the interval $-3 \le x \le 7$. Does the Mean Value Theorem applied on the interval $-3 \le x \le 7$ guarantee a value of c, for $-3 < c < 7$, such that $g''(c)$ is equal to this average rate of change? Why or why not?

a) $\dfrac{g'(7) - g'(-3)}{7-(-3)} = \dfrac{1-(-4)}{10} = \dfrac{1}{2}$

a) $x=1$ g' Δ's inc to dec

$x=4$ g' Δ's dec to inc

b)

x	$g(x)$
-3	$15/2$
2	5
7	$3/2$

$$\int_{-3}^{2} g' = g(2) - g(-3)$$

$$-\tfrac{1}{2}(2)(4) + \tfrac{1}{2}(3)(1) = 5 - g(-3)$$

$$-4 + \tfrac{3}{2} = 5 - g(-3)$$

$$g(-3) = 15/2$$

$$\int_{2}^{7} g' = g(7) - g(2)$$

$$-\tfrac{1}{2}(4)(2) + \tfrac{1}{2}(1)(1) = g(7) - 5$$

$-4 + \frac{1}{2} + 5 = g(7)$

$\frac{3}{2} = g(7)$

Abs Max @ $15/2$ $x = -3$

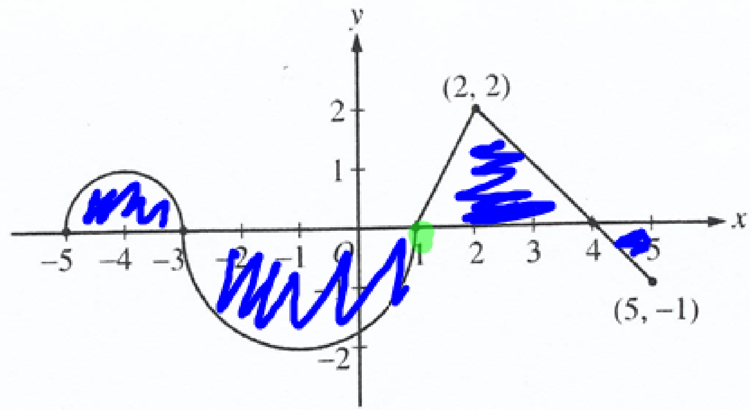
c) ave rate $= \dfrac{g(7) - g(-3)}{7-(-3)}$

Δ

$= \dfrac{\frac{3}{2} - \frac{15}{2}}{10} = -\dfrac{6}{10} = -3/5$

Example 6

e) $(-5,-4) \cup (1,2)$ f' inc and $f' > 0$



Graph of f'

4. Let f be a function defined on the closed interval $-5 \le x \le 5$ with $f(1) = 3$. The graph of f', the derivative of f, consists of two semicircles and two line segments, as shown above.

(a) For $-5 < x < 5$, find all values x at which f has a relative maximum. Justify your answer.

(b) For $-5 < x < 5$, find all values x at which the graph of f has a point of inflection. Justify your answer.

(c) Find all intervals on which the graph of f is concave up and also has positive slope. Explain your reasoning.

(d) Find the absolute minimum value of $f(x)$ over the closed interval $-5 \le x \le 5$. Explain your reasoning.

a) $x = 3, 4$ f' Δ's + to −

b) $x = -4, 2$ f' Δ's inc to dec

$x = -1$ f' Δ's dec to inc

x	$f(x)$
-5	$3+3\pi/2$
1	3
5	$11/2$

$$\int_{-5}^{1} f' = f(1) - f(-5)$$

$$\frac{1}{2}\pi(1)^2 - \frac{1}{2}\pi(2)^2 = 3 - f(-5)$$

$$\frac{\pi}{2} - \frac{4\pi}{2} = 3 - f(-5)$$

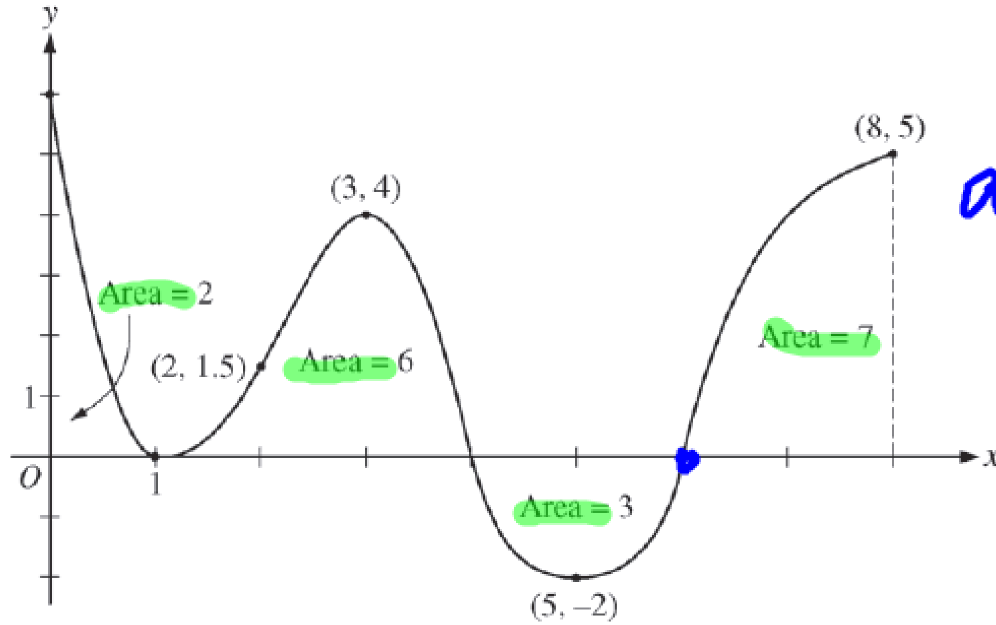
$$f(-5) = 3 + 3\pi/2$$

$$\int_{1}^{5} f' = f(5) - f(1)$$

$$\frac{1}{2}(3)(2) - \frac{1}{2}(1)(1) = f(5) - 3$$

$$11/2 = f(5)$$

Abs min value = 3 when $x = 1$.



Graph of f'

a) $x = 6$
f' Δ's + to −

4. The figure above shows the graph of f', the derivative of a twice-differentiable function f, on the closed interval $0 \le x \le 8$. The graph of f' has horizontal tangent lines at $x = 1$, $x = 3$, and $x = 5$. The areas of the regions between the graph of f' and the x-axis are labeled in the figure. The function f is defined for all real numbers and satisfies $f(8) = 4$.

(a) Find all values of x on the open interval $0 < x < 8$ for which the function f has a local minimum. Justify your answer.

(b) Determine the absolute minimum value of f on the closed interval $0 \le x \le 8$. Justify your answer.

(c) On what open intervals contained in $0 < x < 8$ is the graph of f both concave down and increasing? Explain your reasoning.

(d) The function g is defined by $g(x) = (f(x))^3$. If $f(3) = -\frac{5}{2}$, find the slope of the line tangent to the graph of g at $x = 3$.

x	f(x)
0	-8
6	-3
8	4

$$\int_0^8 f' = f(8) - f(0)$$

$$12 = 4 - f(0)$$

$$f(0) = -8$$

$$\int_6^8 f' = f(8) - f(6)$$

$$7 = 4 - f(6)$$

$$f(6) = -3$$

Abs min $f(0) = -8$

c) $(0,1) \cup (3,4)$ $f' > 0$ and f' dec

d) $g(x) = (f(x))^3$

$g'(x) = 3(f(x))^2 \cdot f'(x)$

$= 3(f(3))^2 \cdot f'(3)$

$= 3(-5/2)^2 \cdot 4$

$= 3\left(\frac{25}{4}\right) \cdot 4 = 75$

a) x=4 IP
f' Δ's dec
to inc.

a) $\int_{-5}^{1} g(x) = f(1) - f(-5)$

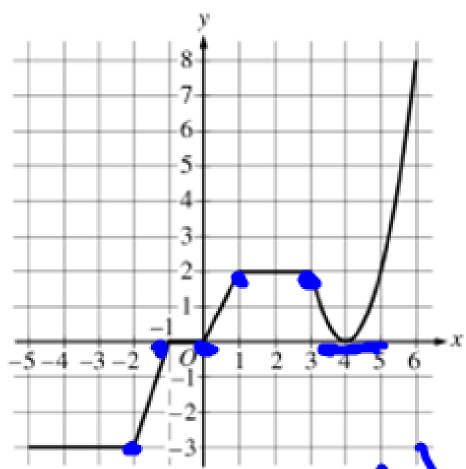
$-9 - \frac{1}{2}(1)(3) + \frac{1}{2}(1)(2)$

$= 3 - f(-5)$

$f(-5) = \frac{25}{2}$

CALCULUS AB
SECTION II, Part B
Time—1 hour
Number of questions—4

NO CALCULATOR IS ALLOWED FOR THESE QUESTIONS.



Graph of g

f'

c) (0,1) ∪ (4,6)
f' > 0 and
f' inc.

3. The graph of the continuous function g, the derivative of the function f, is shown above. The function g is piecewise linear for $-5 \le x < 3$, and $g(x) = 2(x-4)^2$ for $3 \le x \le 6$.

(a) If $f(1) = 3$, what is the value of $f(-5)$?

(b) Evaluate $\int_1^6 g(x)\,dx$.

(c) For $-5 < x < 6$, on what open intervals, if any, is the graph of f both increasing and concave up? Give a reason for your answer.

(d) Find the x-coordinate of each point of inflection of the graph of f. Give a reason for your answer.

$$\int_1^6 g(x)\,dx$$

$$\int_1^3 g(x)\,dx + \int_3^6 2(x-4)^2\,dx$$

$$4 + \frac{2(x-4)^3}{3}\Bigg|_3^6$$

$$4 + \frac{2}{3}\left[(6-4)^3 - (3-4)^3\right]$$

$$4 + \frac{2}{3}(8-(-1)) = 10$$

$g'(x) = f(x)$

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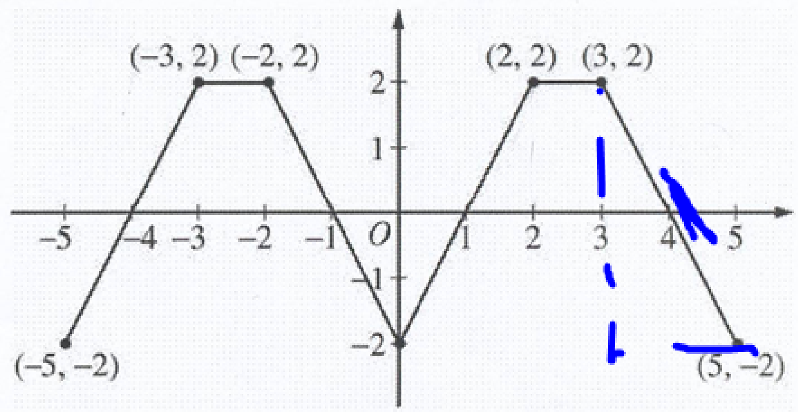
Question 3

g'

The graph of the function f shown above consists of six line segments. Let g be the function given by $g(x) = \int_0^x f(t)\,dt$.

(a) Find $g(4)$, $g'(4)$, and $g''(4)$.

(b) Does g have a relative minimum, a relative maximum, or neither at $x = 1$? Justify your answer.



Graph of f

(c) Suppose that f is defined for all real numbers x and is periodic with a period of length 5. The graph above shows two periods of f. Given that $g(5) = 2$, find $g(10)$ and write an equation for the line tangent to the graph of g at $x = 108$.

a) $g(4) = \int_0^4 f(t)\,dt = (1)(2) + \frac{1}{2}(1)(2)$

$= 2 + 1 = 3$

$g'(4) = 0$

$g''(4) = -2$

b) $x=1$ rel min g' Δ's from − to +.

$\int_0^7 f(t)dt = 2$

Example 2

a) $p'(x) = f(x)$

Let $p(x) = \int_0^x f(t)dt$ and the graph of f is shown in Figure 8.1-4.

$(4, 8)$ $p' < 0$

$p(x) = \int_0^x f(t)dt$

(a) Evaluate: $p(0)$, $p(1)$, $p(4)$

(b) Evaluate: $p(5)$, $p(7)$, $p(8)$

(c) At what value of t does p have a maximum value?

(d) On what interval(s) is p decreasing?

(e) Draw a sketch of the graph of p.

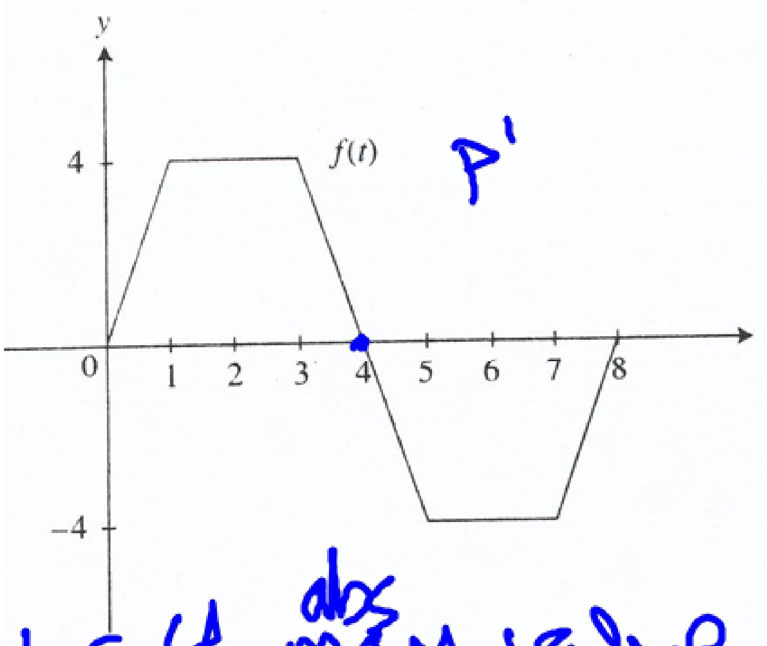
a) b)

x	$p(x)$
0	0
1	2
4	12
5	10
7	2
8	0

$\int_0^1 f(t)dt = \frac{1}{2}(1)(4) = 2$

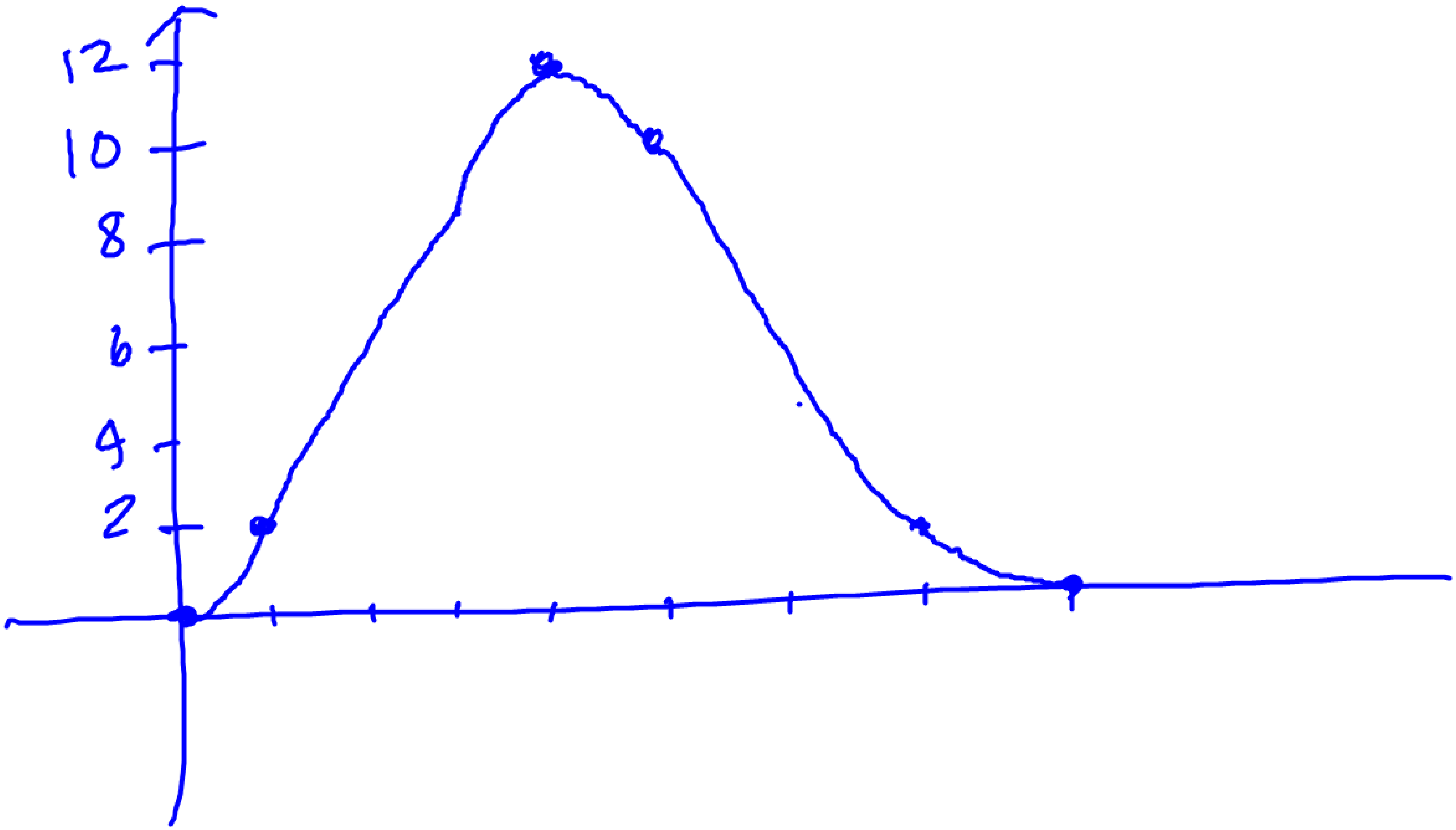
$\int_0^4 = 2(2) + 2(4)$

$\int_0^5 f(t)dt = 2 + 8 = 10$



c) $t = 4$ abs max value

Figure 8.1-4


12
10
8
6
4
2

Assignment Handout

Your Turn!

The position function of a moving particle on a coordinate axis is:

$$s = \int_0^t f(x)dx \text{ feet.}$$

The function f is a differentiable function and its graph is shown below in Figure 8.1-6.

(a) What is the particle's velocity at $t = 4$?
(b) What is the particle's position at $t = 3$?
(c) When is the acceleration zero?
(d) When is the particle moving to the right?
(e) At $t = 8$, is the particle on the right side or left side of the origin?

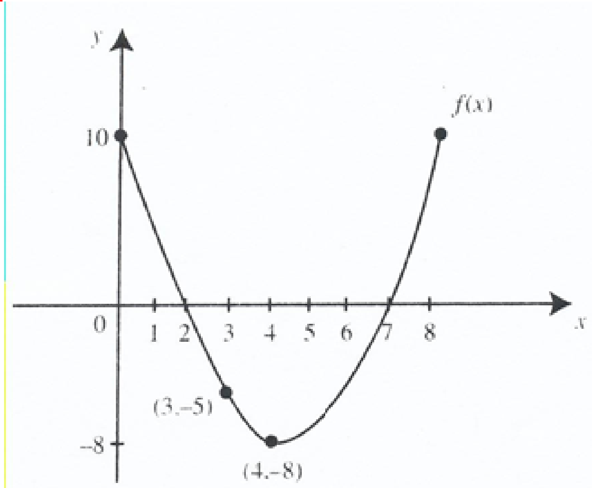
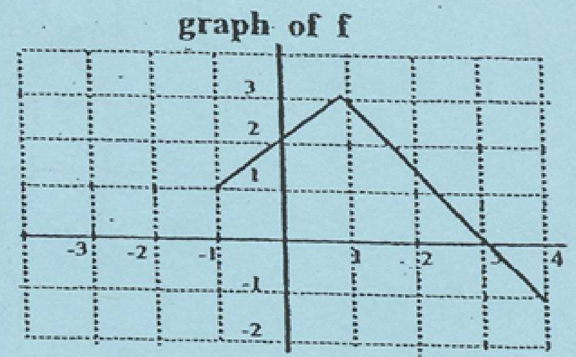


Figure 8.1-6

Functions Defined as Integrals

1. Let $F(x) = \int_{-1}^{x} f(t)\,dt$, $-1 \le x \le 4$, where f is the function graphed in the figure.



a) Complete the following table of values for F.

x	−1	0	1	2	3	4
F(x)						

b) Sketch a graph of F.

Your Turn!

2. Let $G(x) = \int_{-3}^{x} g(t)\,dt$, $-3 \le x \le 3$, where g is the function graphed in the figure.

a) Complete the following table of values for G.

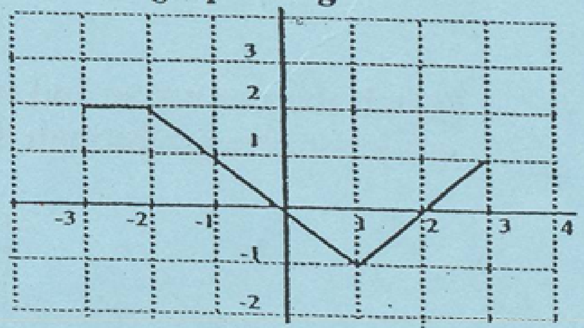
x	−3	−2	−1	0	1	2	3
G(x)							

b) Sketch a graph of G.

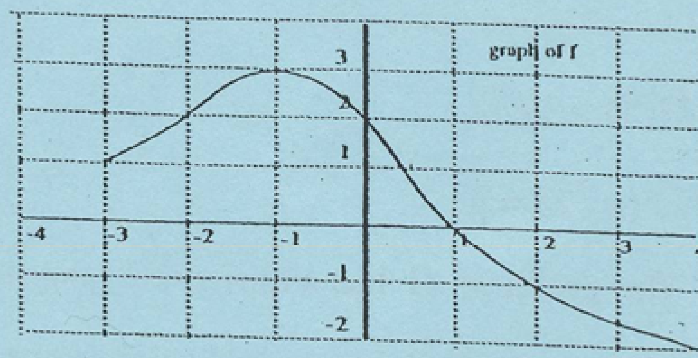
c) Which is larger G(0) or G(1)? Justify your answer.

d) Where is G increasing?

graph of g



6. Let $A(x) = \int_{-3}^{x} f(t)\,dt$, $-3 \le x \le 4$, where f is the function graphed below.



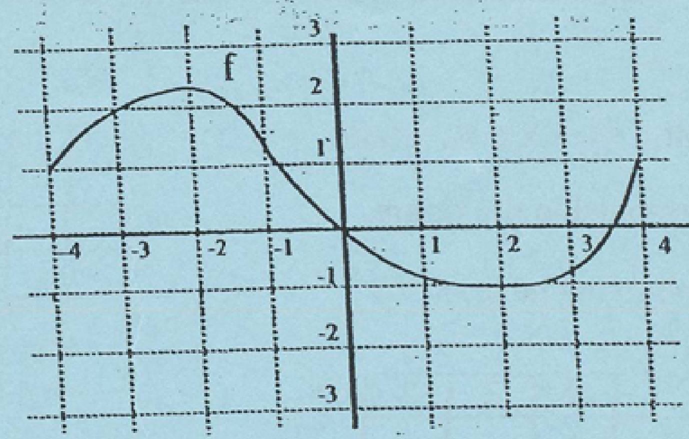
a) Which is larger A(–1) or A(1)? Justify your answer.

b) Which is larger A(2) or A(4) ? Justify your anwer.

c) Where is A increasing?

d) Explain why A has a local maximum at x = 1.

13. The graph of f is shown in the figure.



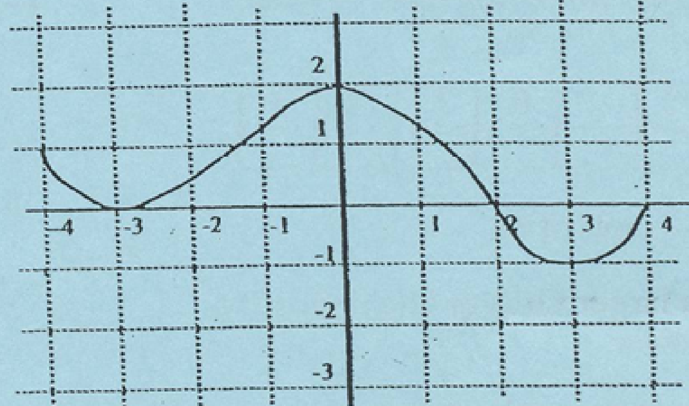
Estimate each of the following:

a) $f'(-1)$

b) $\int_{-4}^{1} f(x)\,dx$

c) $\frac{d}{dx}\left[\int_{-4}^{x} f(t)\,dt\right]$ at $x = -2$

14. Sketch the derivative and an antiderivative of the function in the figure. Make the antiderivative go through the origin.



21. A function G is defined by $G(x) = \int_0^x \sqrt{1+t^2}\, dt$ for all real numbers x. Determine whether the following statements are true or false. Justify your answers.

a) G continuous at $x = 0$.

b) $G(3) > G(1)$

c) $G'(2\sqrt{2}) = 3$

d) The graph of G has a horizontal tangent at $x = 0$.

e) The graph of G has an inflection point at $(0, 0)$.

Assignment Handouts

Great Practice Problems

2. The approximate *average* rate of change of the function $f(x) = \int_0^x \sin(t^2)\,dt$ over the interval [1, 3] is

(A) 0.19 (B) 0.23 (C) 0.27 (D) 0.31 (E) 0.35

CALC

8. The average rate of change of the function $f(x) = \int_0^x \sqrt{1 + \cos(t^2)}\,dt$ over the interval [1, 3] is nearest to

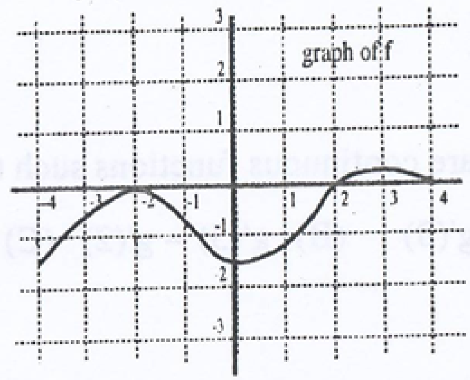
CALC

(A) 0.85
(B) 0.86
(C) 0.87
(D) 0.88
(E) 0.89

15. The graph of the function f is shown at the right. If the function G is defined by

$G(x) = \int_{-4}^{x} f(t)\,dt$, for $-4 \le x \le 4$, which of the following statements about G are true?

graph of the function f



I. G is increasing on (1, 2)

II. G is decreasing on (−4, −3)

III. G(0) < 0

(A) None (B) II only (C) III only (D) II and III only (E) I and II only

17. Consider the function F defined so that $F(x) + 5 = \int_{2}^{x} \sin\left(\frac{\pi t}{4}\right) dt.$

CALC

The value of $F(2) + F'(2)$ is

(A) 0

(B) 1

(C) $\frac{\pi}{4}$

(D) 4

(E) −4

4. If $F(x) = \int_1^x (\cos 6t + 1)\ dt$, then $F'(x) =$

(A) $\sin 6x + x$ (B) $\cos 6x + 1$ (C) $\frac{1}{6}\sin 6x + x$

(D) $-\frac{1}{6}\sin 6x + 1$ (E) $\sin 6x + 1$

12. Suppose $F(x) = \int_0^{x^2} \frac{1}{2 + t^3}\ dt$ for all real x, then $F'(-1) =$

(A) 2 (B) 1 C) $\frac{1}{3}$ (D) -2 (E) $-\frac{2}{3}$

12. Suppose $F(x) = \int_0^{\cos x} \sqrt{1+t^3}\, dt$ for all real x, then $F'\left(\frac{\pi}{2}\right) =$

(A) -1

(B) 0

(C) $\frac{1}{2}$

(D) 1

(E) $\frac{\sqrt{3}}{2}$

17. If f and g are continuous functions such that $g'(x) = f(x)$ for all x, then $\int_2^3 f(x)\, dx =$

(A) $g'(2) - g'(3)$ (B) $g'(3) - g'(2)$ (C) $g(3) - g(2)$ (D) $f(3) - f(2)$ (E) $f'(3) - f'(2)$

27. If the function G is defined for all real numbers by $G(x) = \int_0^{2x} \cos(t^2)\, dt$, then $G'(\sqrt{\pi}) =$

(A) 2 (B) 1 (C) 0 (D) −1 (E) −2

28. If for all $x > 0$, $G(x) = \int_1^{x} \sin(\ln 2t)\, dt$, then the value of $G''\left(\frac{1}{2}\right)$ is

(A) 0

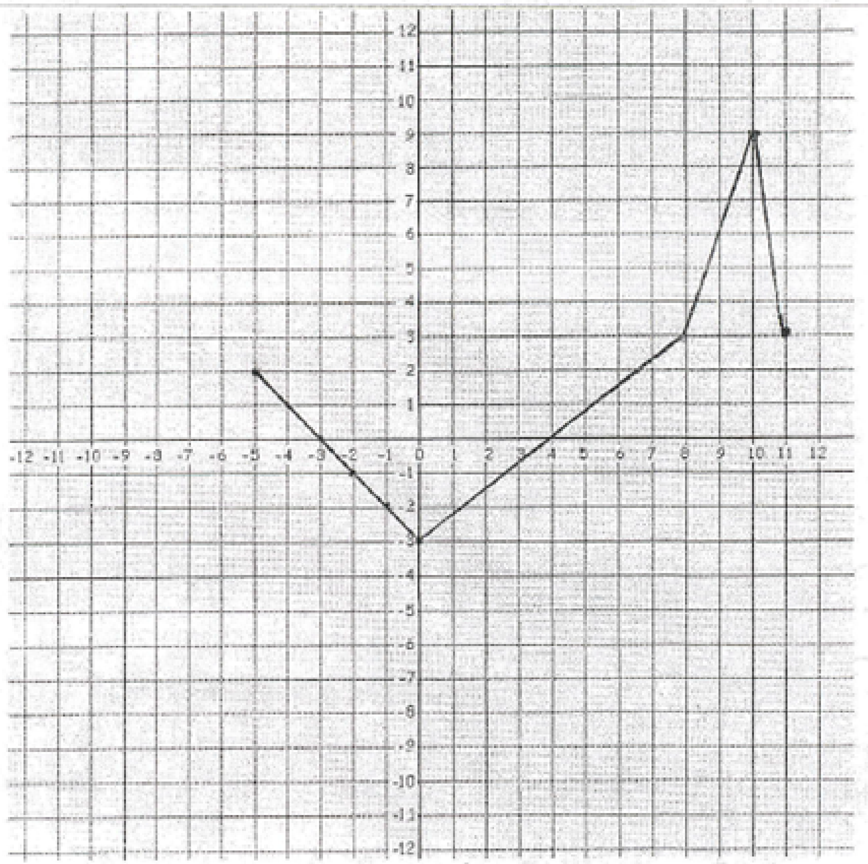
(B) $\frac{1}{2}$

(C) 1

(D) 2

(E) undefined

1. Given that $g(x)=\int_{2} f(t)dt$, and the graph of f is given below:



a) Find: $g(-4)$ $g'(-4)$ $g''(-4)$

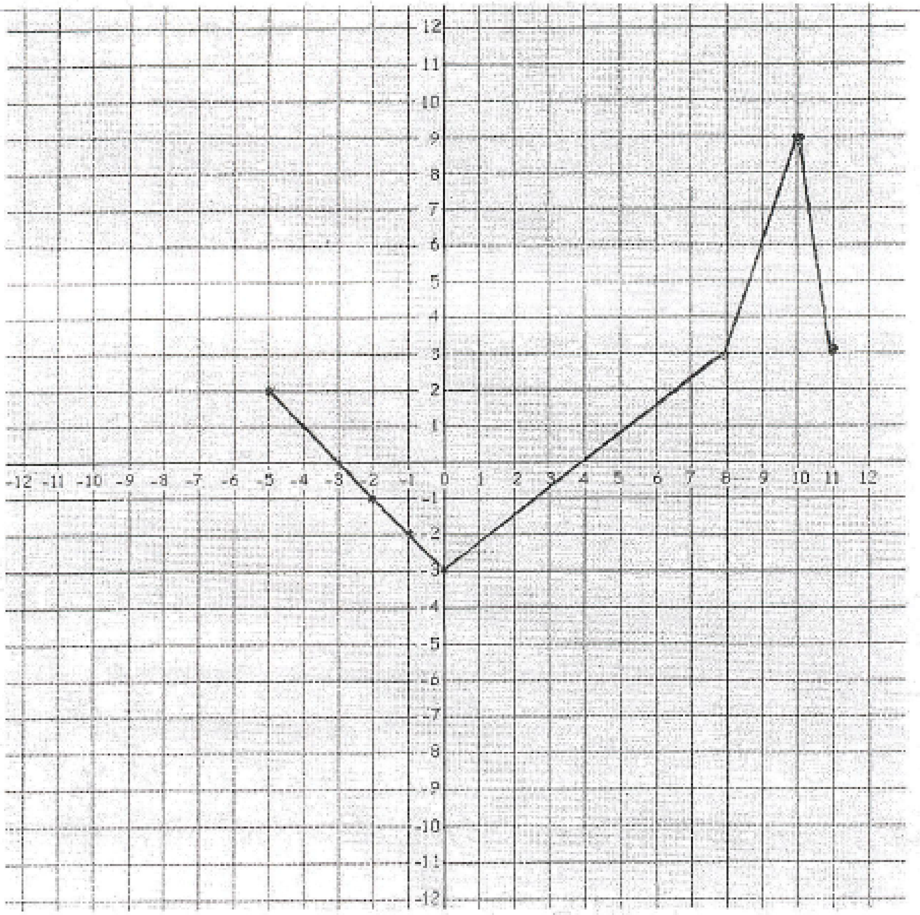
b) Find the equation of the tangent line to $g(x)$ at $x=-4$

c) Over what interval is $g(x)$ increasing? Explain why?

d) Over what interval is $g(x)$ concave down? Explain why?

e) Where does $g(x)$ have inflection points? Explain why?

2. Let f be a function defined on the closed interval $[-5,11]$ with $f(4)=2$.
The graph of f' is given below.



a) Where does f have a relative maximum? Explain why.

b) Where does f have a relative minimum? Explain why.

c) Find the value of $f(-4)$

d) Find the value of $f(8)$

2. Suppose you know that a certain function f is twice differentiable and that its graph over $[0,2]$ is given in the figure below. As you see, the printer was sloppy and spilled a lot of ink on the graph. Decide, if possible, whether each of the following definite integrals is positive, equal to 0, or negative.

a. $\displaystyle\int_0^2 f''(x)\,dx$ b. $\displaystyle\int_0^2 f'(x)\,dx$ c. $\displaystyle\int_0^2 f(x)\,dx$