5.4 Fundamental Theorem of Calculus

5.4 The Fundamental Theorem of Calculus.

Learning Targets:

1. SWBAT use the FTC part one to find derivatives of functions defined as integrals.

2 SWBAT to use the FTC part two to evaluate integrals.

3. SWBAT apply both parts of the FTC to solve a number of AP exam questions.

The Fundamental Theorem of Calculus consists of two parts.

Theorem 4 **The Fundamental Theorem of Calculus, Part 1**

If f is continuous on $[a, b]$, then the function

$$F(x) = \int_a^x f(t)\,dt$$

has a derivative at every point x in $[a, b]$, and

$$\frac{dF}{dx} = \frac{d}{dx}\int_a^x f(t)\,dt = f(x).$$

It says **every continuous function** is the **derivative of some other function.**

It says the process of **integration** and **differentiation** are **inverses of each other.**

$$\text{Ex. 1a) Find } \frac{d}{dx}\int_{-\pi}^{x} \cos t\,dt$$

Limits are a constant to x!

$$= \cos x$$

$$\text{b) Find } \frac{d}{dx}\int_{0}^{x} \frac{1}{1+t^2}\,dt \quad = \frac{1}{1+x^2}$$

Ex.2 Find $\dfrac{d}{dx}\displaystyle\int_x^2 2t^2\,dt$

$$= \frac{d}{dx} - \int_2^x 2t^2\,dt$$

$$= -2x^2$$

Ex.3 Find $\frac{dy}{dx}$ if $y = \int_1^{x^2} \cos t\,dt$

Limits not a constant to x!

$2x(\cos x^2)$

$$\text{Ex.4 Find } \frac{dy}{dx} \text{ of } y = \int_{2x}^{x^2} \frac{1}{2+e^t}\,dt$$

$$\left(\frac{1}{2+e^{x^2}}\right)2x - \frac{1}{2+e^{2x}} \cdot 2$$

$$\frac{2x}{2+e^{x^2}} - \frac{2}{2+e^{2x}}$$

2003 AP MC Question

23. $\frac{d}{dx}\left(\int_0^{x^2} \sin\left(t^3\right) dt\right) =$

(A) $-\cos\left(x^6\right)$ (B) $\sin\left(x^3\right)$ (C) $\sin\left(x^6\right)$ (D) $2x\sin\left(x^3\right)$ (E) $2x\sin\left(x^6\right)$

$= \left(\sin(x^2)^3\right) 2x$

$= 2x \sin x^6$

b) $(-1, 1)$ $g' > 0$

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c) $(0, 2)$ g' dec

Question 4

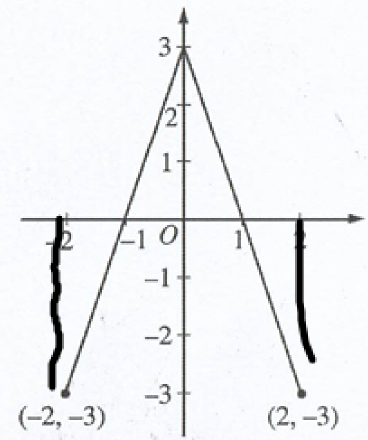
The graph of the function f shown above consists of two line segments. Let g be the function given by $g(x) = \int_0^x f(t)\,dt$.

(a) Find $g(-1)$, $g'(-1)$, and $g''(-1)$.

(b) For what values of x in the open interval $(-2, 2)$ is g increasing? Explain your reasoning.

(c) For what values of x in the open interval $(-2, 2)$ is the graph of g concave down? Explain your reasoning.

(d) On the axes provided, sketch the graph of g on the closed interval $[-2, 2]$.



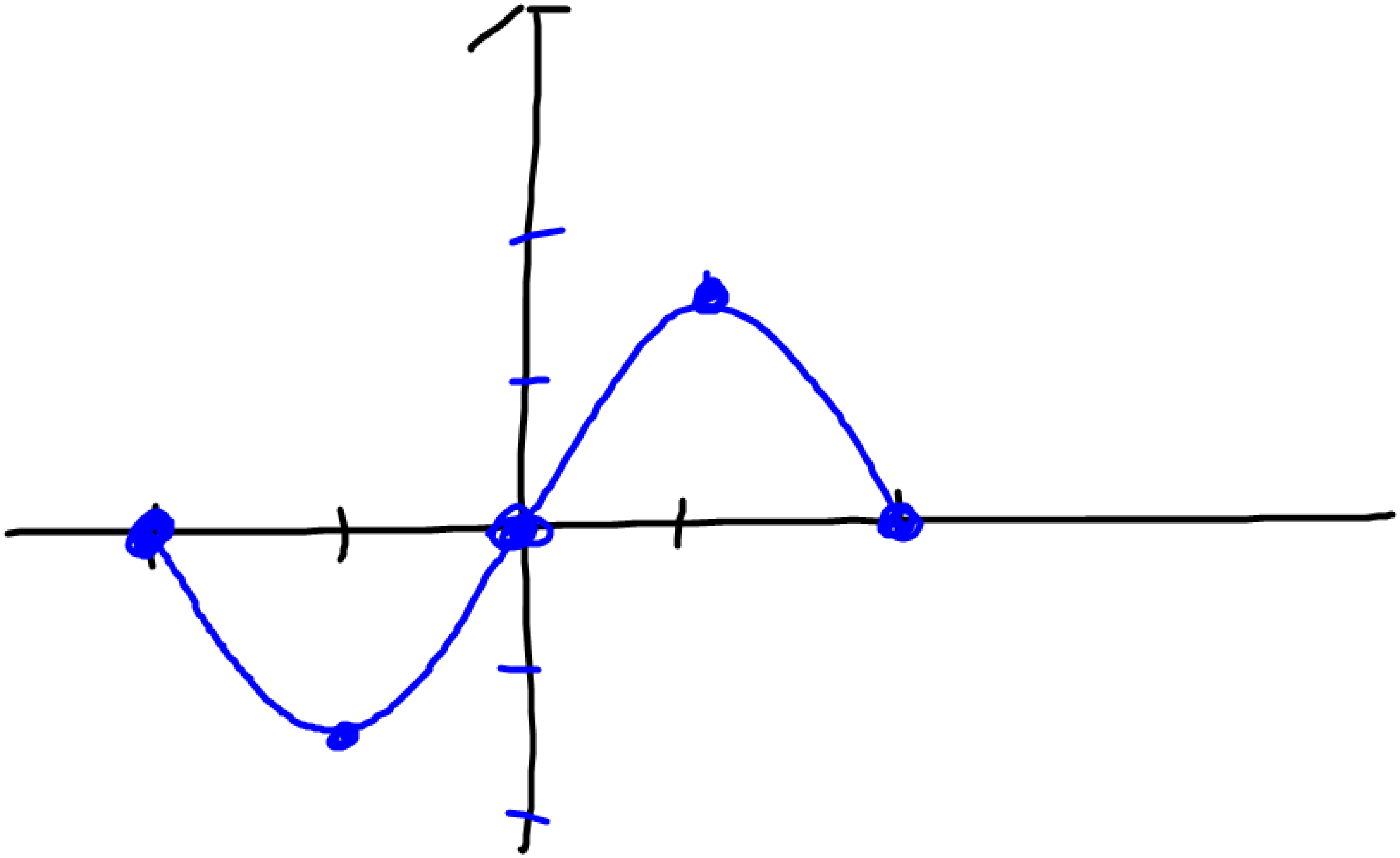
Graph of f

g'

a) $g(-1) = \int_0^{-1} f(t)\,dt = -\int_{-1}^{0} f(t)\,dt$

$= -\left(\frac{1}{2}(1)(3)\right)$

$= -3/2$



$$g(x) = \int_0^x f(t)\,dt$$

x	$g(x)$
-2	0
-1	$-3/2$
0	0
1	$3/2$
2	0

$$g(-2) = \int_0^{-2} f(t)\,dt$$

$$= -\int_{-2}^{0} f(t)\,dt$$

$$g(1) = \int_0^1 f(t)\,dt$$

$$g(2) = \int_0^2 f(t)\,dt$$

$$g'(-1) = 0$$

$$g(x) = \int_0^x f(t)\,dt$$

$$g'(x) = f(x)$$

$$g''(-1) = 3$$

d) $x=1$ is an IP $\quad g'$ Δ's inc to dec.

AB–5 / BC–5 $\qquad$ 1999

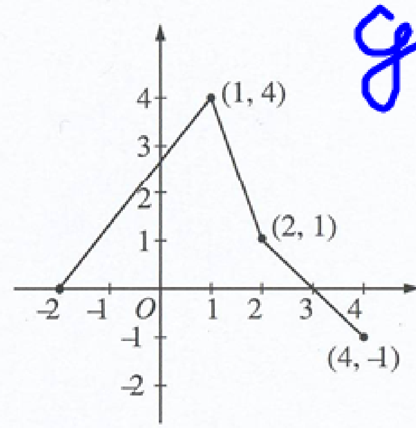
5. The graph of the function f, consisting of three line segments, is given above. Let $g(x) = \int_1^x f(t)\,dt$.

   (a) Compute $g(4)$ and $g(-2)$.

   (b) Find the instantaneous rate of change of g, with respect to x, at $x = 1$.

   (c) Find the absolute minimum value of g on the closed interval $[-2, 4]$. Justify your answer.

   (d) The second derivative of g is not defined at $x = 1$ and $x = 2$. How many of these values are x–coordinates of points of inflection of the graph of g? Justify your answer.



g'

a) $g(4) = \int_1^4 f(t)\,dt = \frac{1}{2}(1)(4+1) = 5/2$

$g(-2) = \int_1^{-2} f(t)\,dt = -\int_{-2}^{1} f(t)\,dt$

$= -\left(\frac{1}{2}(3)(4)\right) = -6$

b)

$$g'(x) = f(x)$$

$$g'(1) = 4$$

c) Abs min's occur at endpoints or where $g'(x)$ Δ's from $-$ to $+$.

Only possibility are endpoints.

$g(-2) = 5/2$

$g(4) = -6$

Min value -6.

$g(0) = \int_2^0 f(t)\,dt$

$= -\int_0^2 f(t)\,dt$

c) 2 b/c $y = 7/3$ crosses graph g' twice.

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Question 5

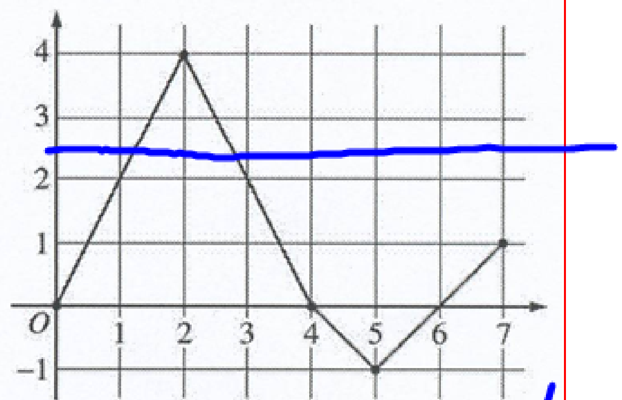
Let f be a function defined on the closed interval $[0,7]$. The graph of f, consisting of four line segments, is shown above. Let g be the function given by $g(x) = \int_2^x f(t)\,dt$.

(a) Find $g(3)$, $g'(3)$, and $g''(3)$.

(b) Find the average rate of change of g on the interval $0 \le x \le 3$.

(c) For how many values c, where $0 < c < 3$, is $g'(c)$ equal to the average rate found in part (b)? Explain your reasoning.

(d) Find the x-coordinate of each point of inflection of the graph of g on the interval $0 < x < 7$. Justify your answer.



Graph of f

g'

a) $g(3) = \int_2^3 f(t)\,dt = \frac{1}{2}(1)(4+2) = 3$

$g'(3) = f(3) = 2$

$g''(3) = -2$

b) $\dfrac{g(3) - g(0)}{3 - 0}$

$= \dfrac{3+4}{3} = 7/3$

d) IP $x=2$

g' Δ's inc to dec

IP $x=5$

g' Δ's dec to inc

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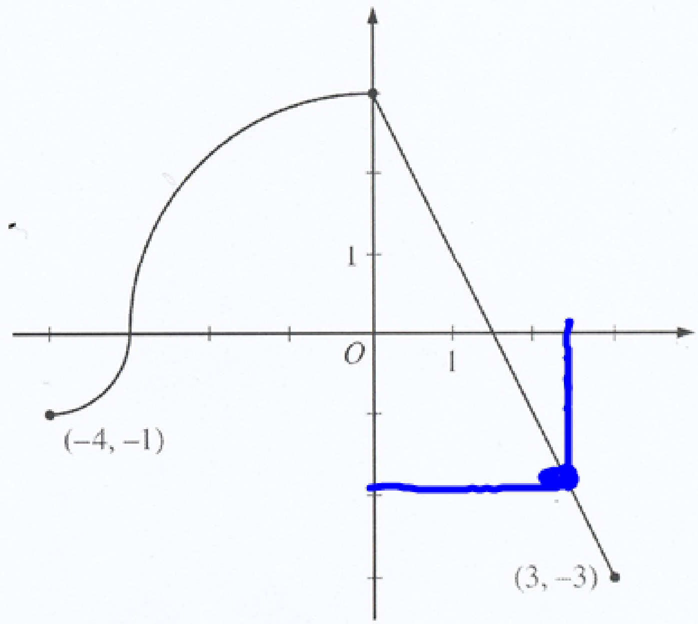
Question 4

The continuous function f is defined on the interval $-4 \le x \le 3$. The graph of f consists of two quarter circles and one line segment, as shown in the figure above.



Graph of f

Let $g(x) = 2x + \int_0^x f(t)\,dt$.

(a) Find $g(-3)$. Find $g'(x)$ and evaluate $g'(-3)$.

(b) Determine the x-coordinate of the point at which g has an absolute maximum on the interval $-4 \le x \le 3$. Justify your answer.

(c) Find all values of x on the interval $-4 < x < 3$ for which the graph of g has a point of inflection. Give a reason for your answer.

(d) Find the average rate of change of f on the interval $-4 \le x \le 3$. There is no point c, $-4 < c < 3$, for which $f'(c)$ is equal to that average rate of change. Explain why this statement does not contradict the Mean Value Theorem.

a) $g(-3) = 2(-3) + \int_{0}^{-3} f(t)\,dt$

$$= -6 - \int_{-3}^{0} f(t)\,dt$$

$$= -6 - \left(\frac{\pi(3)^2}{4}\right)$$

$$= -6 - \frac{9\pi}{4}$$

$g'(x) = 2 + f(x)$

$g'(-3) = 2 + f(-3) = 2$

b) $g'(x) = 2 + f(x)$

$2 + f(x) = 0$

$f(x) = -2$

x	$g(x)$
-4	$-8-2\pi$
$5/2$	6.5
3	6

$$g(-4) = 2(-4) + \int_0^{-4} f(t)\,dt$$

$$= -8 - \int_{-4}^{0} f(t)\,dt$$

$$= -8 - \left(\frac{-\pi(1)^2}{4} + \frac{9\pi}{4}\right)$$

$$= -8 - 2\pi$$

$$g(5/2) = 2\left(\frac{5}{2}\right) + \int_0^{5/2} f(t)\,dt$$

$$= 5 + \text{[illegible]} + \frac{1}{2}\left(\frac{3}{2}\right)(3)$$

$$- \frac{1}{2}\left(\frac{3}{2}\right)(2)$$

$$= 5 + \frac{9}{4} - \frac{3}{4}$$

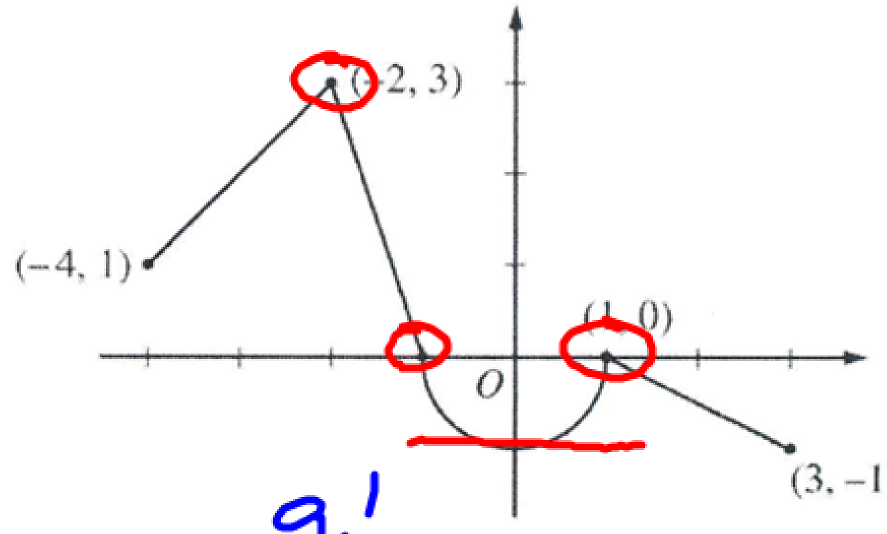
$$= 5 + \frac{3}{2}$$

$$g(3) = 2(3) + \int_0^3 f(t)\,dt$$

$$= 6 + 0$$

Question 3

Let f be the continuous function defined on $[-4, 3]$ whose graph, consisting of three line segments and a semicircle centered at the origin, is given above. Let g be the function given by $g(x) = \int_1^x f(t)\,dt$.



Graph of f

(a) Find the values of $g(2)$ and $g(-2)$.

(b) For each of $g'(-3)$ and $g''(-3)$, find the value or state that it does not exist.

(c) Find the x-coordinate of each point at which the graph of g has a horizontal tangent line. For each of these points, determine whether g has a relative minimum, relative maximum, or neither a minimum nor a maximum at the point. Justify your answers.

(d) For $-4 < x < 3$, find all values of x for which the graph of g has a point of inflection. Explain your reasoning.

a) $g(2) = \int_1^2 f(t)\,dt$

$= -\left(\frac{1}{2}(1)\left(\frac{1}{2}\right)\right) = -\frac{1}{4}$

$g(-2) = \int_1^{-2} f(t)\,dt$

$= -\int_{-2}^{1} f(t)\,dt$

$= -\left(\frac{1}{2}(1)(3) - \frac{\pi}{2}(1)^2\right)$

$= -\frac{3}{2} + \pi/2$

b) $g'(-3)$ $g''(-3)$

$g'(x) = f(x)$

$g'(-3) = 2$

$g''(-3) = 1$

c)

$g'(x) = 0$

at $x = 1$ or $x = -1$

$x = {}^{-}1$ rel max b/c g' Δ's + to −.

$x = 1$ neither b/c g' does not change sign at $x = -1$.

$g''(x) = 0$ at $x = 0$

$g''(x)$ undefined at $x = -2, -1, 1$

$x = -2$ IP b/c g' Δ's inc to dec

$x = 0$ IP b/c g' Δ's dec to inc

$x = 1$ IP b/c g' Δ's inc to dec

Assignment

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