5.4 Concavity and the Second Derivative

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Learning Targets:

1. SWBAT find intervals of concavity using the second derivative test.
2. SWBAT find inflection points using the second derivative test.
3. SWBAT find relative extrema using the seconsd derivative test.


recall

A function is said to be **increasing** if the graph is **“going uphill”.**

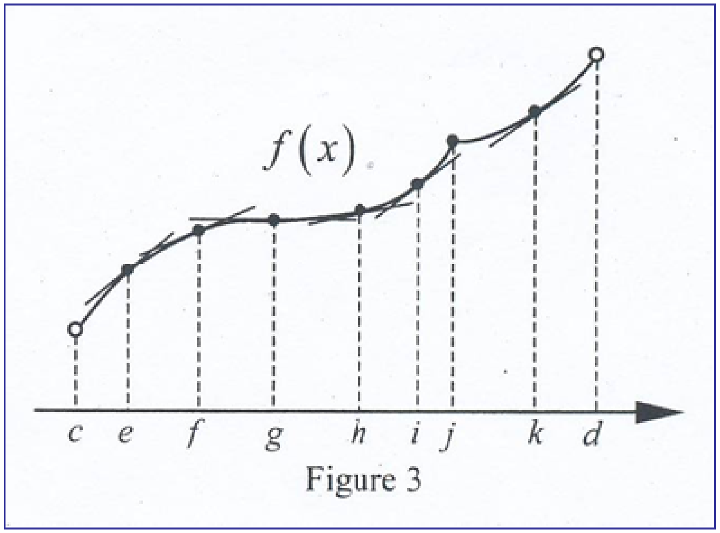


Figure 3

If $f'(x) > 0$ for all x in an open interval, then $f(x)$ is increasing on this open interval.

A function is said to be **decreasing** if the graph is **"going downhill".**

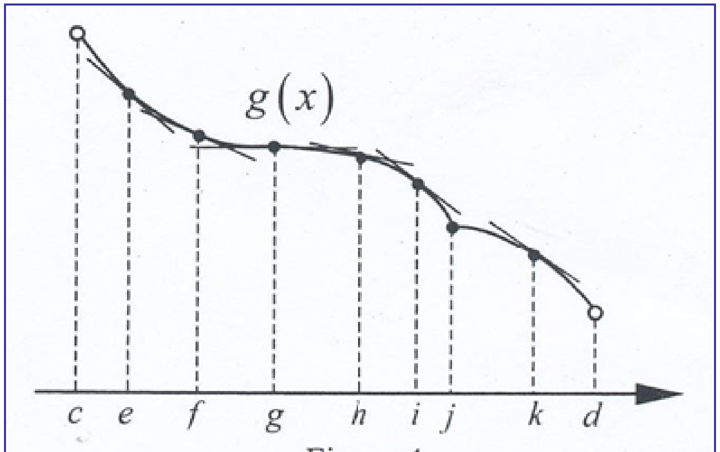


Figure 4

If $f'(x) < 0$ for all x in an open interval, then $f(x)$ is decreasing on this open interval.

A function is said to be **concave up** if the graph is shaped like a **"valley".**

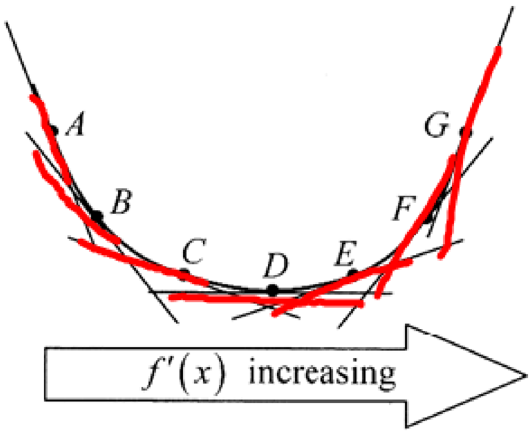


Figure 1

$f'(x)$ is increasing, thus $f''(x)$ is positive. The curve is concave up.

$f''(x) > 0$

$\boxed{f' \text{ inc}}$

CU

If $f''(x) > 0$, for all x in an interval I, then $f(x)$ is concave up.

A function is said to be **concave down** if the graph is shaped like a **"hill".**

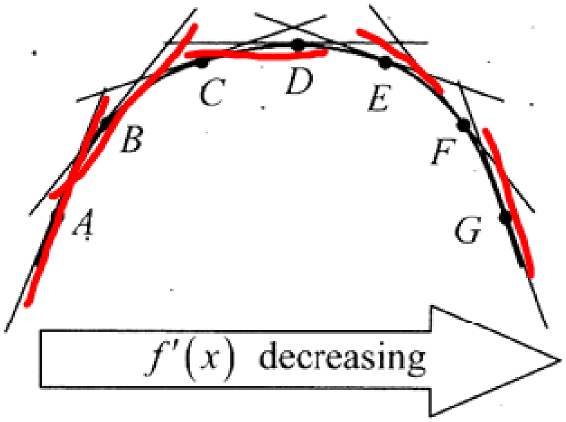


Figure 2

$f'(x)$ is decreasing, thus $f''(x)$ is negative. The curve is concave down.

$f'' < 0$

$\boxed{f' \text{ dec}}$

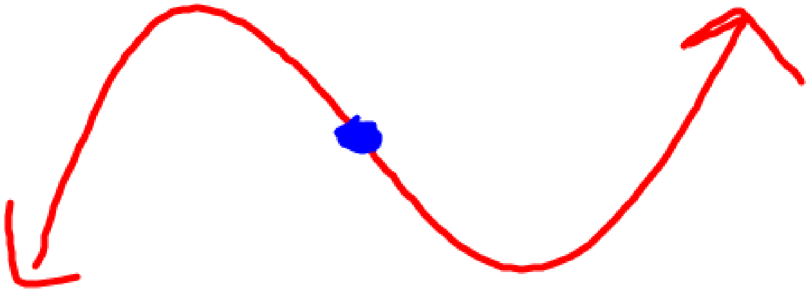
CD

If $f''(x) < 0$, for all x in an interval I, then $f(x)$ is concave down.

Inflection Points

Inflection Points or **IP's** are points where the concavity of a function changes.

If inflection points exist, they will occur where $\boxed{f''(x) = 0}$ or where $\boxed{f''(x) \text{ does not exist.}}$



Ex.1 Determine where the following functions are concave up, concave down and find all IP's

a) $y = x^3 - 3x^2 + 4x - 5$

$y' = 3x^2 - 6x + 4$

$y'' = 6x - 6$

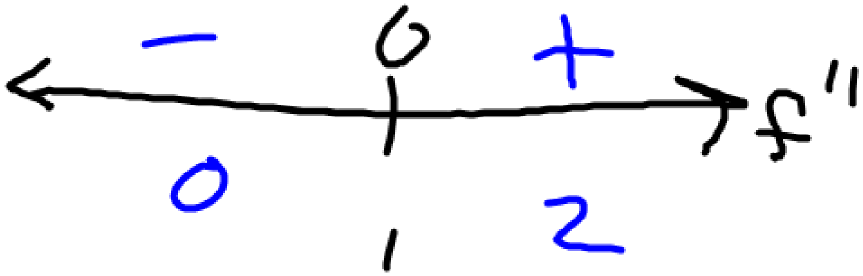
$\underline{y'' = 0}$

$6x - 6 = 0$

$6x = 6$

$\boxed{x = 1}$

$y\cancel{''\infty}$



$\underline{CD}$

$(-\infty, 1)$

$\underline{CU}$

$(1, \infty)$

$\underline{IP}$

$f(1) = -3$

b) $y = \dfrac{x}{x^2+1}$

$$y' = \frac{(x^2+1)(1) - x(2x)}{(x^2+1)^2}$$

$$y' = \frac{1-x^2}{(x^2+1)^2}$$

$$y'' = \frac{(x^2+1)^{\cancel{2}}(-2x) - (1-x^2)\cdot 2\cancel{(x^2+1)}^{1}\cdot 2x}{(x^2+1)^{\cancel{4}\,3}}$$

$$= \frac{-2x^3 - 2x - 4x + 4x^3}{(x^2+1)^3}$$

$$y'' = \frac{2x^3 - 6x}{(x^2+1)^3} = \frac{2x(x-\sqrt{3})(x+\sqrt{3})}{(x^2+1)^3}$$

↑

$\times$

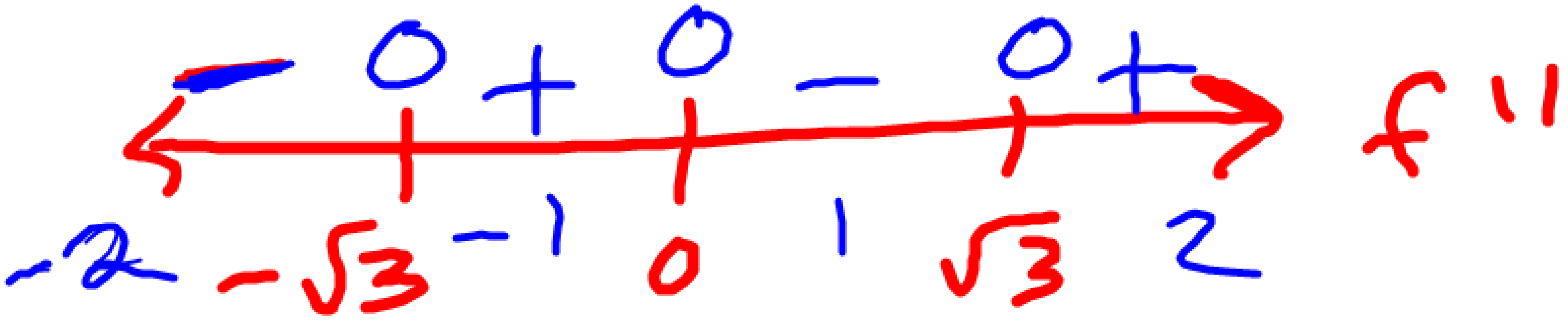
$\underline{y'' = 0}$

$2x^3 - 6x = 0$

$2x(x^2 - 3) = 0$

$2x(x-\sqrt{3})(x+\sqrt{3}) = 0$

$x = 0 \qquad x = \sqrt{3} \qquad x = -\sqrt{3}$



CD

$(-\infty,-\sqrt{3})\cup(0,\sqrt{3})$

CU

$(-\sqrt{3},0)\cup(\sqrt{3},\infty)$

IP

$f(-\sqrt{3}) = -\frac{\sqrt{3}}{4}$

$f(0) = 0$

$f(\sqrt{3}) = \frac{\sqrt{3}}{4}$

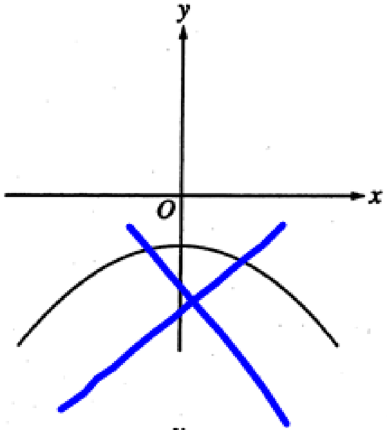
$f(x) < 0$

$f'(x) < 0$

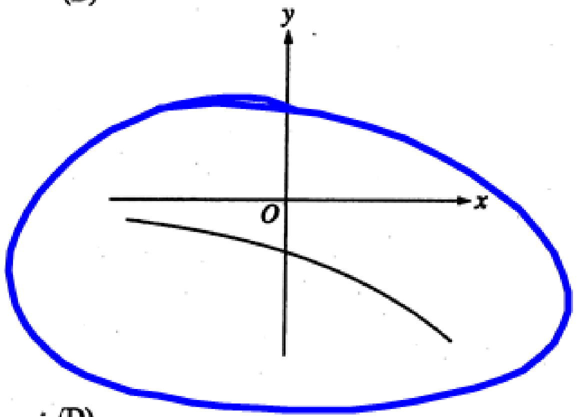
$f''(x) < 0$

10. The function f has the property that $f(x)$, $f'(x)$, and $f''(x)$ are negative for all real values x. Which of the following could be the graph of f ?

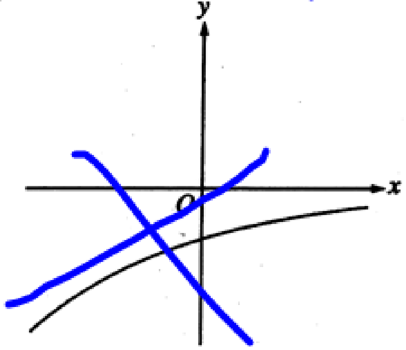
(A)



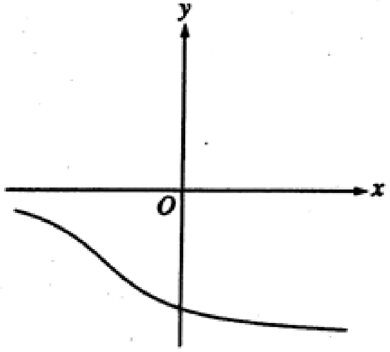
(B)



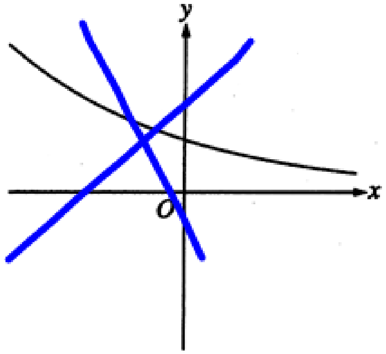
(C)

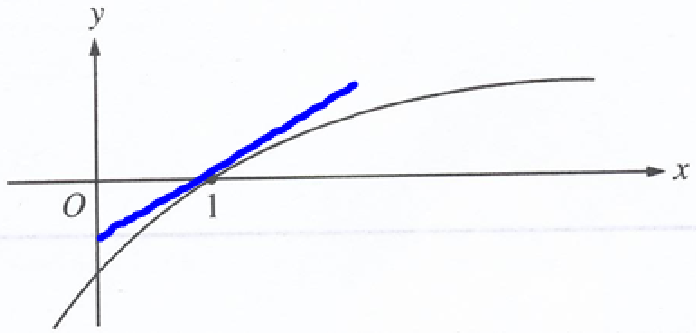


(D)



(E)





17. The graph of a twice-differentiable function f is shown in the figure above. Which of the following is true?

(A) $f(1) < f'(1) < f''(1)$

(B) $f(1) < f''(1) < f'(1)$

(C) $f'(1) < f(1) < f''(1)$

[✓ (D)] $f''(1) < f(1) < f'(1)$

(E) $f''(1) < f'(1) < f(1)$

$f(1) = 0$

$f'(1) > 0$

$f''(1) < 0$

Assignment

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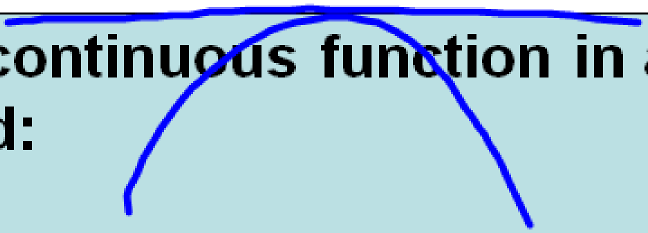
#’s 1 a-f,

4,5,6,8,11,12,14

Second Derivative Test For Local Maximums and Local Minimums

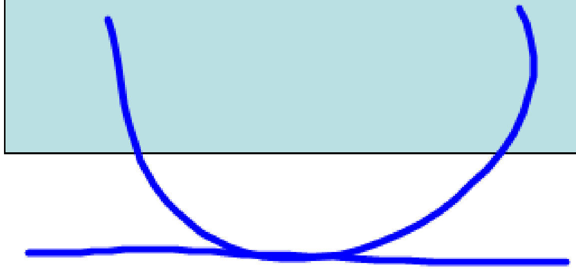
We can use the **second derivative** to also help us find **local maximums** and **minimums.**

Suppose *f(x)* is a continuous function in an interval containing *"c"* and:



$$f'(c) = 0 \text{ and } f''(c) < 0, \text{ then "c" is a local maximum.}$$

$$f'(c) = 0 \text{ and } f''(c) > 0, \text{ then "c" is a local minimum.}$$



Ex.2 Use the second derivative test to determine any local maximums and minimums.

$$f(x)=\frac{1}{4}x^4-x^3$$

$f' = x^3 - 3x^2$

$\underline{f' = 0}$

$x^3 - 3x^2 = 0$

$x^2(x-3) = 0$

$\boxed{x=0}$ OR $\boxed{x=3}$

$f'' = 3x^2 - 6x$

$f''(3) = 3(3)^2 - 6(3) > 0$

∴ $x = 3$.

rel min

$f''(0) = 3(0)^2 - 6(0) = 0$

Neither

1. Where is f(x) increasing?

(1, 3) ∪ (4, 6)

2. Where is f(x) decreasing?

(0, 1) ∪ (3, 4) ∪ (6, 10)

3. Where is f(x) concave up?

(0, 2) ∪ (4, 8)

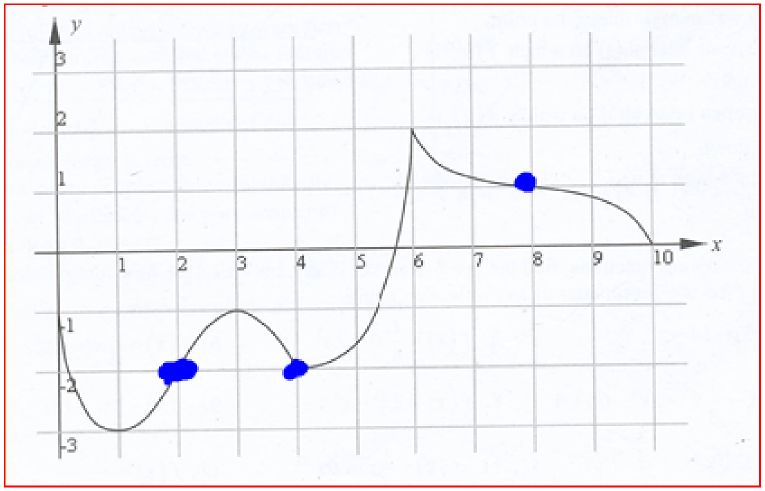
4. Where is f(x) concave down?

(2, 4) ∪ (8, 10)

5. What are the x coordinates of the inflection points of f(x)?

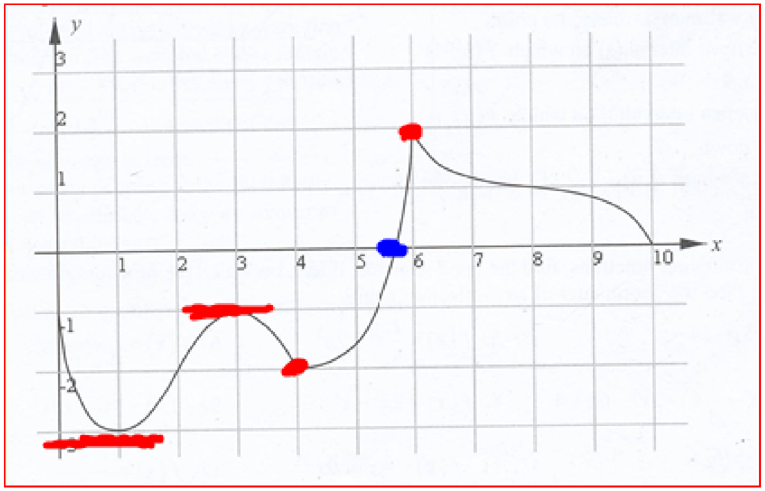
x = 2, 4, 8

Graph of $f(x)$



Justify

* Graph of $f'(x)$



1. Where is f(x) increasing?

$(5.5, 10)$ $f'(x) > 0$

2. Where is f(x) decreasing?

$(0, 5.5)$ $f'(x) < 0$

3. Where is f(x) concave up?

$(1,3) \cup (4,6)$ f' inc

4. Where is f(x) concave down?

$(0,1) \cup (3,4) \cup (6,10)$ f' dec

5. What are the x coordinates of the inflection points of f(x)?

$x = 1, 4$ f' Δ's dec to inc

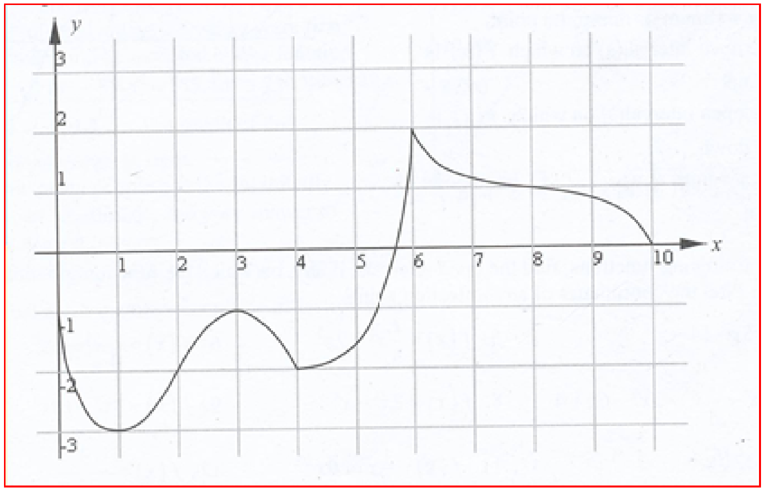
$x = 3, 6$ f' Δ's inc to dec

6. What are the x coordinates of any relative max and mins?

$x = 5.5$

rel min f' Δ's $-$ to $+$.

Graph of $f''(x)$



1. Where is f(x) concave up?

$(5.5, 10)$

$f'' > 0$

2. Where is f(x) concave down?

$(0, 5.5)$ $f'' < 0$

3. What are the x coordinates of the inflection points of f(x)?

$x = 5.5$ f'' Δ's from $-$ to $+$.

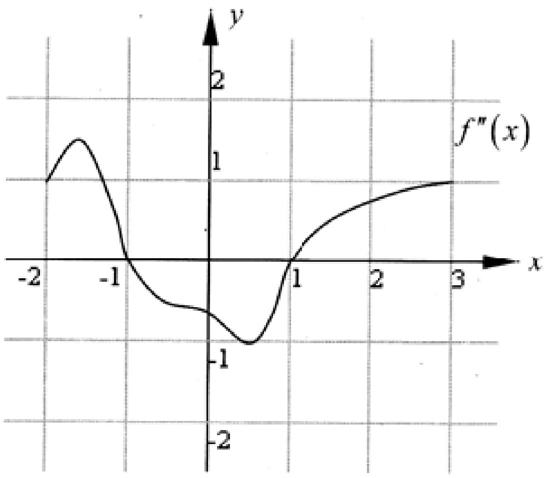
Assignment Pages 246-247
#'s 2,3,16,17,18,19

2. Shown at right is the graph of $f''(x)$ on the interval $[-2,3]$. Answer the following questions about $f(x)$. Explain your reasoning.

(a) State the open interval(s) on which $f(x)$ is concave up.

(b) State the open interval(s) on which $f(x)$ is concave down.

(c) State the x-value(s), if any, at which $f(x)$ has an inflection point.



Question 2

3. Shown below right is the graph of $f'(x)$ on the interval $[-3,3]$. Answer the following questions about $f(x)$. Explain your reasoning.

(a) State the open interval(s) on which $f(x)$ is increasing.

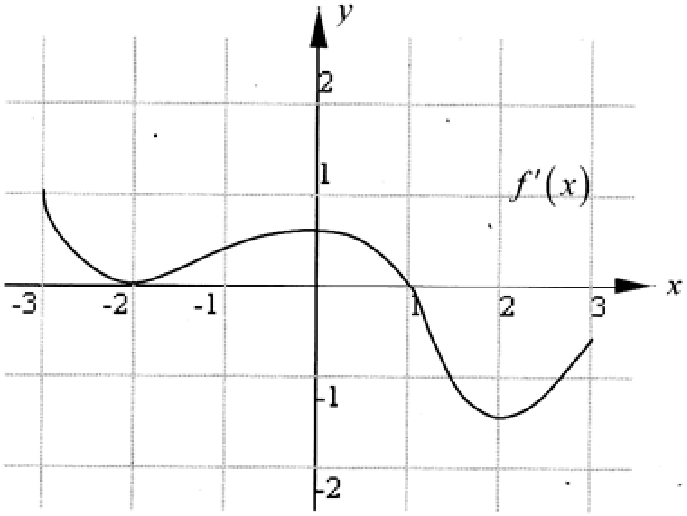
(b) State the open interval(s) on which $f(x)$ is decreasing.

(c) State the x-value(s), if any, at which $f(x)$ has a relative maximum or minimum point.

(d) State the open interval(s) on which $f(x)$ is concave up.

(e) State the open interval(s) on which $f(x)$ is concave down.

(f) State the x-values at which $f(x)$ has a point of inflection.



Question 3