5.3 Trigonometric Integrals

Trigonometric Integrals Involving Sine and Cosine

$$\int \sin^M x \cos^N x \, dx$$

1. If the power of the Sine is odd, save one Sine factor, and convert the remaining factors to Cosine.

2. If the power of the Cosine is odd, save one Cosine factor, and convert the remaining factors to Sine.

3. If the powers of the Sine and Cosine are both even, then convert using the half angle identities:

$$\sin^2 x = \frac{1}{2}\left(1 - \cos(2x)\right) \qquad \cos^2 x = \frac{1}{2}\left(1 + \cos(2x)\right)$$

Example: $\int cos^3 x dx$

$$\cos^2 x = 1 - \sin^2 x$$

$$\int \cos x \cos^2 x \, dx$$

$$\int \cos x \left(1 - \sin^2 x\right) dx = \int (1 - u^2) \, du$$

let $u = \sin x$

$du = \cos x \, dx$

$$= u - \frac{u^3}{3} + C$$

$$= \sin x - \frac{\sin^3 x}{3} + C$$

Example: $\int sin^5 x cos^2 x dx$

$$\int \sin x\left(\sin^2 x\right)^2 \cos^2 x\, dx$$

$\sin^2 x = 1-\cos^2 x$

$$\int \sin x\left(1-\cos^2 x\right)^2 \cos^2 x\, dx$$

let $u = \cos x$

$du = -\sin x dx$

$-du = \sin x dx$

$$= -\int (1-u^2)^2 u^2\, du$$

$$= -\int (1-2u^2+u^4)u^2\, du$$

$$= -\int \left(u^2 - 2u^4 + u^6\right) du$$

$$= -\left[\frac{u^3}{3} - \frac{2}{5}u^5 + \frac{u^7}{7} + C\right]$$

$$= -\frac{\cos^3 x}{3} + \frac{2}{5}\cos^5 x - \frac{\cos^7 x}{7} + C$$

Example: $\int sin^2 x dx$

$$= \int \frac{1}{2}(1 - \cos 2x)\,dx$$

$$= \frac{1}{2}\int (1 - \cos 2x)\,dx$$

$$= \boxed{\frac{1}{2}\left[x - \frac{\sin 2x}{2}\right] +} \; C$$

$$= \frac{1}{2}\left[x - \frac{\cancel{2}\sin x \cos x}{\cancel{2}}\right] + C$$

$$= \frac{1}{2}x - \frac{1}{2}\sin x \cos x + C$$

Example: $\int_0^{\pi} sin^2 x dx$

$$= \frac{1}{2}\left[x - \frac{\sin 2x}{2}\right]\Bigg|_0^{\pi}$$

$$= \frac{1}{2}\left[(\pi - 0) - (0)\right]$$

$$= \pi/2$$

Example: $\int cos^3(2x) sin^6(2x) dx$

$$\int \sin^6(2x)\cos^2(2x)\cos(2x)\,dx$$

$$1-\sin^2(2x)$$

$$\int \sin^6(2x)(1-\sin^2(2x))\underbrace{\cos 2x\,dx}$$

let $u = \sin 2x$

$\frac{1}{2}du = \cancel{2}\cos 2x\,dx$

$$= \frac{1}{2}\int u^6(1-u^2)\ du$$

$$= \frac{1}{2}\int u^6 - u^8\ du$$

$$= \frac{1}{2}\left[\frac{u^7}{7} - \frac{u^9}{9}\right] + C$$

$$= \frac{u^7}{14} - \frac{u^9}{18} + C$$

$$= \frac{\sin^7(2x)}{14} - \frac{\sin^9(2x)}{18} + C$$

Example: $\int sin^4 x dx$

$$\int (\sin^2 x)^2 dx$$

$$\int \left(\frac{1}{2}(1-\cos 2x)\right)^2 dx$$

$$= \frac{1}{4}\int (1-2\cos 2x + \cos^2 2x) dx$$

$$= \frac{1}{4}\int \left(1-2\cos 2x + \frac{1}{2}(1+\cos 4x)\right) dx$$

$$= \frac{1}{4}\int \left(1-2\cos 2x + \frac{1}{2} + \frac{1}{2}\cos 4x\right) dx$$

$$= \frac{1}{4}\int \left(\frac{3}{2} - 2\cos 2x + \frac{1}{2}\cos 4x \right) dx$$

$$= \frac{1}{4}\left[\frac{3}{2}x - \sin 2x + \frac{1}{8}\sin 4x \right] + C$$

Trigonometric Integrals Involving Secant and Tangent

$$\int \sec^{M} x \ \tan^{N} x \ dx$$

1. If the power of the Secant is even, save a Secant-squared factor, and convert the remaining factors to Tangent.

2. If the power of the Tangent is odd, save a Secant-Tangent factor, and convert the remaining factors to Secant.

Example: $\int tan^6 x sec^4 x dx$

$$\int \tan^6 x\,(\sec^2 x)(\sec^2 x)\,dx$$

$$\int \tan^6 x\,(1+\tan^2 x)\sec^2 x\,dx$$

let $u = \tan x$

$du = \sec^2 x\,dx$

$$= \int u^6(1+u^2)\,du$$

$$= \int (u^6+u^8)\,du$$

$$= \frac{u^7}{7} + \frac{u^9}{9} + C$$

$$= \frac{\tan^7 x}{7} + \frac{\tan^9 x}{9} + C$$

Example: $\int tan^5 x sec^7 x dx$

$$\int ((\tan^2 x)^2 \sec^6 x \ \sec x \tan x)\, dx$$

$$\int (\sec^2 x - 1)^2 \sec^6 x \ \sec x \tan x)\, dx$$

let $u = \sec x$

$du = \sec x \tan x\, dx$

$$\int (u^2-1)^2 u^6\, du = \int (u^4 - 2u^2 + 1) u^6\, du$$

$$\int (u^{10} - 2u^8 + u^6)\,du$$

$$= \frac{u^{11}}{11} - \frac{2u^9}{9} + \frac{u^7}{7} + C$$

$$= \frac{\sec^{11}x}{11} - \frac{2\sec^9 x}{9} + \frac{\sec^7 x}{7} + C$$

Other integrals that come into play:

$$\int tanx = ln|secx| + C$$

$$\int secx = ln|secx + tanx| + C$$

Example: $\int tan^3 x dx$

$$\int \tan x \tan^2 x \, dx$$

$$\int \tan x (\sec^2 x - 1) \, dx$$

$$\int (\tan x \sec^2 x - \tan x) \, dx$$

$$\int \tan x \sec^2 x \, dx - \underline{\int \tan x \, dx}$$

Let $u = \tan x$

$du = \sec^2 x \, dx$

$$\int u\,du - \int \tan x\,dx$$

$$= \frac{u^2}{2} - \ln|\sec x| + C$$

$$= \frac{\tan^2 x}{2} - \ln|\sec x| + C$$

Assignment: Handout