5.3 Increasing and Decreasing Intervals

The First Derivative Test

5.3 Increasing and Decreasing Intervals

Learning Targets:

1. SWBAT find intervals of increase and decrease using the first derivative test.
2. SWBAT find relative extrema using the first derivative test..
3. SWBAT find relationships between the graph of a function and the graph of its derivative.

A function is said to be **increasing** if the graph is **“going uphill”.**

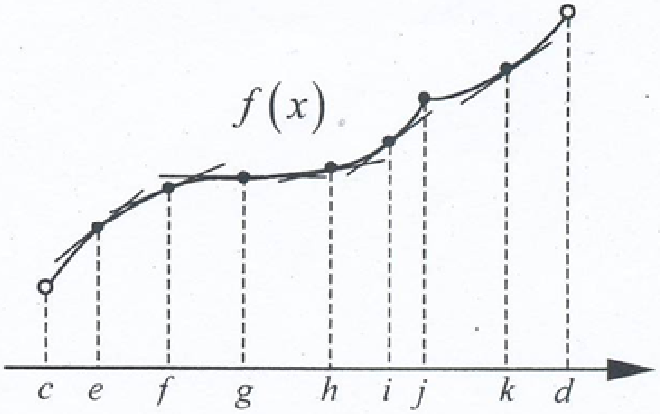


Figure 3

What do we notice about the slopes of the tangent lines for a function that is increasing?

tangent slopes +

If $f'(x) > 0$ for all x in an open interval, then $f(x)$ is increasing on this open interval.

A function is said to be **decreasing** if the graph is **"going downhill".**

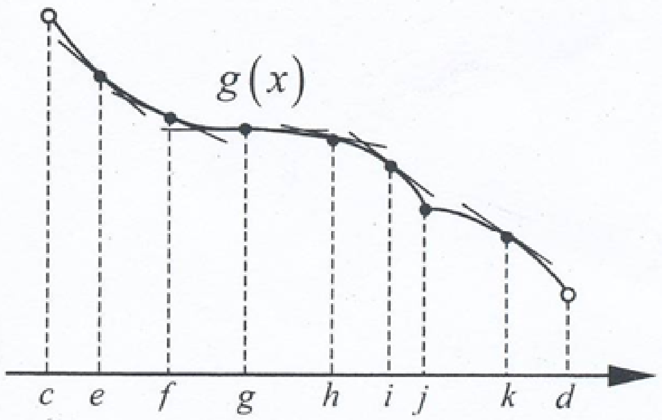


Figure 4

What do we notice about the slopes of the tangent lines for a function that is decreasing?

slopes -

If $f'(x) < 0$ for all x in an open interval, then $f(x)$ is decreasing on this open interval.

Recall!

2. Critical Numbers

Figures 1 through 4 illustrate that if a function $f(x)$ has a relative extremum at $x = c$, then either $f'(c) = 0$ or $f'(c)$ does not exist. If $x = c$ is in the domain of $f(x)$ and $f'(c) = 0$ or $f'(c)$ does not exist, then $x = c$ is said to be a **critical number** of $f(x)$.

Thus if $f(x)$ has a relative extremum at $x = c$, then $x = c$ is a critical number of $f(x)$. Note that $x = c$ is a critical number in Figures 5 and 6 at right.

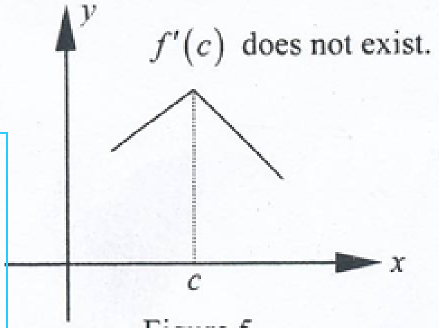


Figure 5

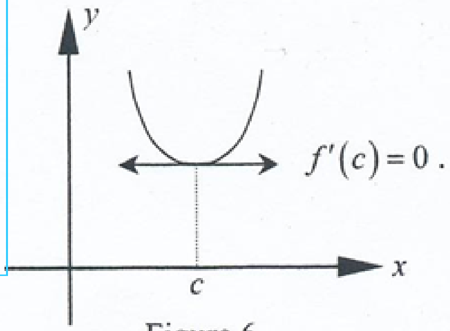
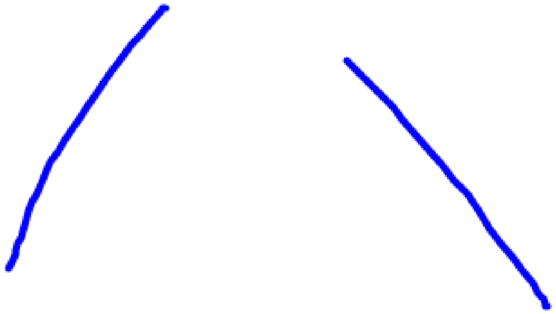


Figure 6

The First Derivative Test

If c is a critical number of a continuous function $f(x)$, then:

- $f(x)$ has a relative minimum at $x = c$ if $f'(x)$ switches signs from negative to positive at c.
- $f(x)$ has a relative maximum at $x = c$ if $f'(x)$ switches signs from positive to negative at c.



Ex.1 Find the open intervals for which the following function is increasing and decreasing. In addition find any relative extrema.

$$f(x)=\frac{1}{3}x^3+\frac{1}{2}x^2-12x+4$$

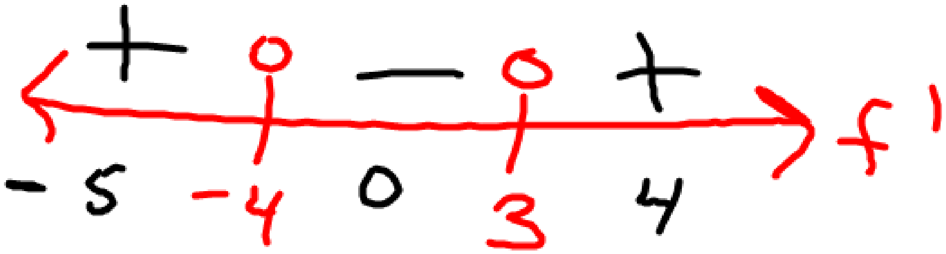
$f'(x) = x^2+x-12$

$\underline{f'(x)=0}$

$x^2+x-12=0$

$(x+4)(x-3)=0$

$x=-4$ or $x=3$



inc

$(-\infty,-4)\cup(3,\infty)$

dec

$(-4,3)$

Rel Max

$$f(-4) = \frac{1}{3}(-4)^3 + \frac{1}{2}(-4)^2 - 12(-4) + 4$$

$$= -\frac{64}{3} + 8 + 48 + 4$$

$$= -\frac{64}{3} + 60$$

$$= -\frac{64}{3} + \frac{180}{3} = \frac{116}{3} \approx 38.7$$

<u>Rel Min</u>

$$f(3) = \frac{1}{3}(3)^3 + \frac{1}{2}(3)^2 - 12(3) + 4$$

$$= 9 + 9/2 - 36 + 4 = 9/2 - 23 = -37/2$$

Ex.2 Find the open intervals for which the following function is increasing and decreasing. In addition find any relative extrema.

$x \neq 2$

$$f(x)=\frac{x^2}{x-2}$$

$$\overset{2x^2-4x}{}$$

$$f'=\frac{\cancel{(x-2)(2x)}-x^2}{(x-2)^2}$$

$$f'=\frac{x^2-4x}{(x-2)^2}$$

$$=\frac{\boxed{x(x-4)}}{(x-2)^2}$$

$\underline{f'=0}$

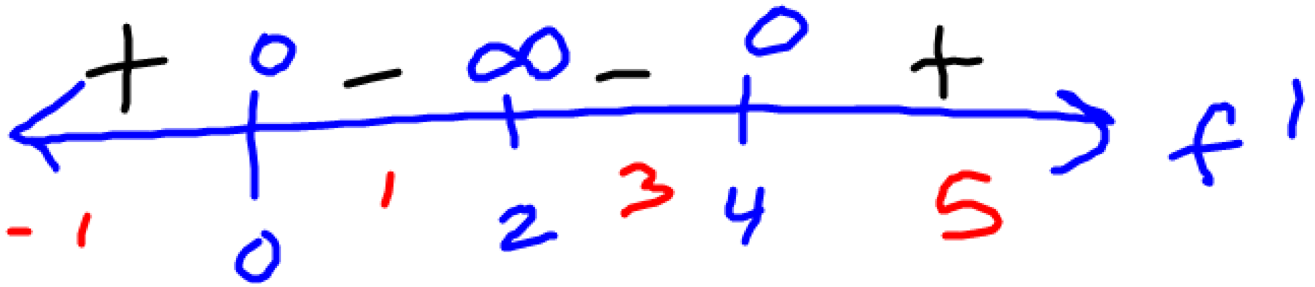
$x^2-4x=0$

$x(x-4)=0$

$\boxed{x=0}$ $\boxed{x=4}$

$\underline{f'\ \infty}$

$\boxed{x=2}$



inc

$(-\infty, 0) \cup (4, \infty)$

Rel Max

$f(0) = 0$

Rel Min

$f(4) = \frac{(4)^2}{4-2} = 8$

Dec

$(0,2) \cup (2,4)$

$(0,4)$ but $x \neq 2$

Ex.3 Find the open intervals for which the following function is increasing and decreasing. In addition find any relative extrema.

$$f(x) = 3x^{\frac{1}{3}} - \frac{3}{2}x^{\frac{2}{3}}$$

$$f'(x) = x^{-2/3} - x^{-1/3}$$

$$= \frac{1}{x^{2/3}} - \frac{1}{x^{1/3}} \cdot \frac{x^{1/3}}{x^{1/3}}$$

$$= \frac{1}{x^{2/3}} - \frac{x^{1/3}}{x^{2/3}} = \boxed{\frac{1 - x^{1/3}}{x^{2/3}}}$$

$$\frac{1-(-1)^{1/3}}{(-1)^{2/3}}$$

$$+$$

$$+$$

$$\frac{1-\left(\frac{1}{8}\right)^{1/3}}{+}$$

$$\frac{1-(8)^{1/3}}{+}$$

$\underline{f' = 0}$

$1 - x^{1/3} = 0$

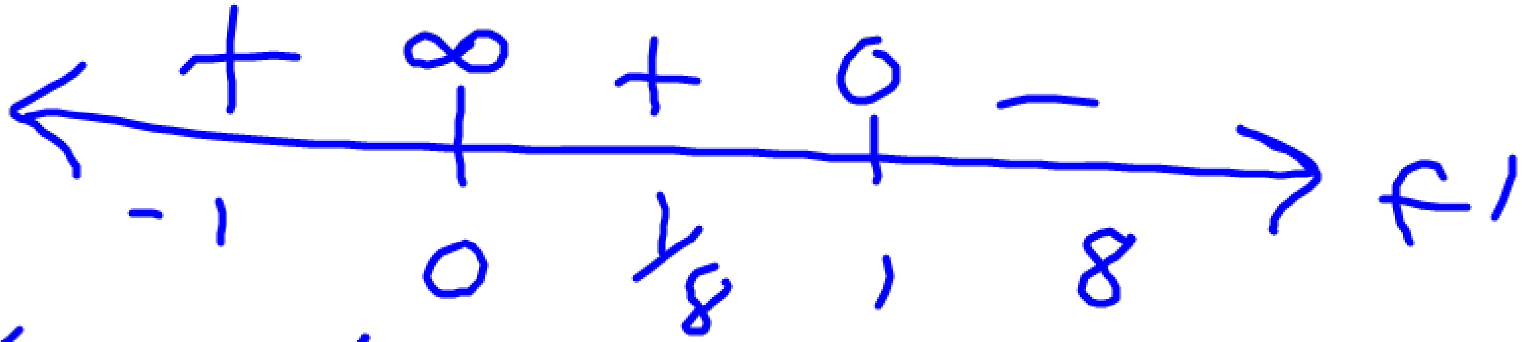
$(1)^3 = (x^{1/3})^3$

$\boxed{1 = x}$

$\underline{f' \infty}$

$x^{2/3} = 0$

$\boxed{x = 0}$



$\underline{\text{inc}}$

$(-\infty, 1)$

$\underline{\text{dec}}$

$(1, \infty)$

$\underline{\text{Rel Max}}$

$f(1) = 3 - \frac{3}{2} = \boxed{\frac{3}{2}}$

Ex.4 Find the open intervals for which the following function is increasing and decreasing. In addition find any relative extrema.

$$\boxed{D: x \geq -3}$$

$$f(x) = x\sqrt{x+3} = x(x+3)^{1/2}$$

$$f' = x \cdot \frac{1}{2}(x+3)^{-1/2} + (x+3)^{1/2}(1)$$

$$= \frac{x}{2(x+3)^{1/2}} + (x+3)^{1/2} \quad \frac{2(x+3)^{1/2}}{2(x+3)^{1/2}}$$

$$= \frac{x}{2(x+3)^{1/2}} + \frac{\overset{2x+6}{\cancel{2(x+3)}}}{2(x+3)^{1/2}}$$

$$= \frac{3x+6}{2(x+3)^{1/2}}$$

$f' = 0$

$3x + 6 = 0$

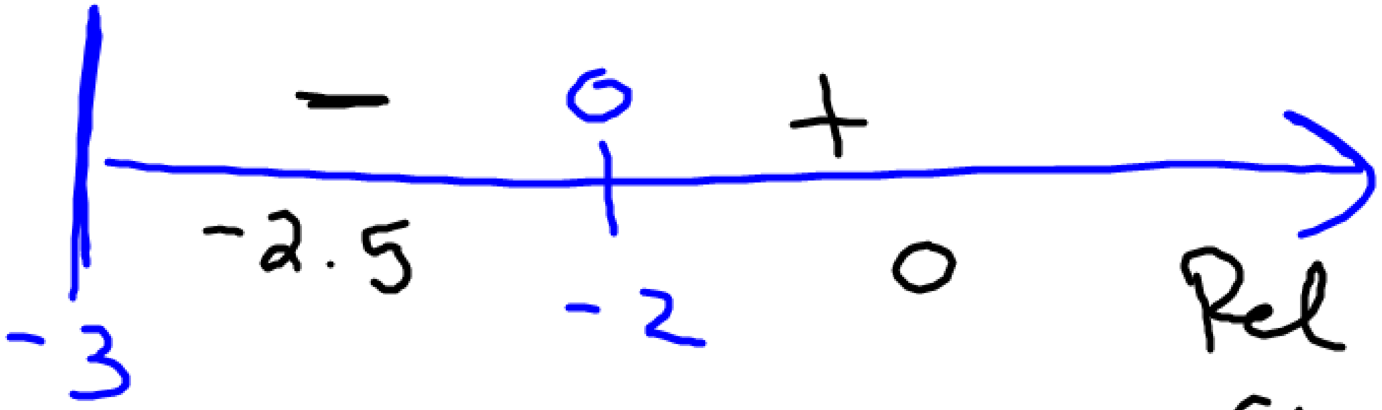
$3x = -6$

$\boxed{x = -2}$

$f'\ \infty$

$2(x+3)^{1/2} = 0$

$\boxed{x = -3}$



Rel Min

$f(-2) = 0$

Dec

$(-3, -2)$

Inc

$(-2, \infty)$

Assignment

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#’s 2,3,6,7,8,11,13,15,16,17,18

⑰

$$f(x) = \overset{x^{1/2}}{\sqrt{x}}\,(6 - x^{1})$$

D: $x \geq 0$

$$f(x) = 6x^{1/2} - x^{3/2}$$

$$f'(x) = 3x^{-1/2} - \frac{3}{2}x^{1/2}$$

$$= \frac{{}^{2}\,3}{2\,x^{1/2}} - \frac{3}{2}x^{1/2}\,\frac{x^{1/2}}{x^{1/2}}$$

$$= \frac{6}{2x^{1/2}} - \frac{3x}{2x^{1/2}} = \boxed{\frac{6-3x}{2x^{1/2}}}$$

$$\underline{f' = 0}$$

$$6 - 3x = 0$$

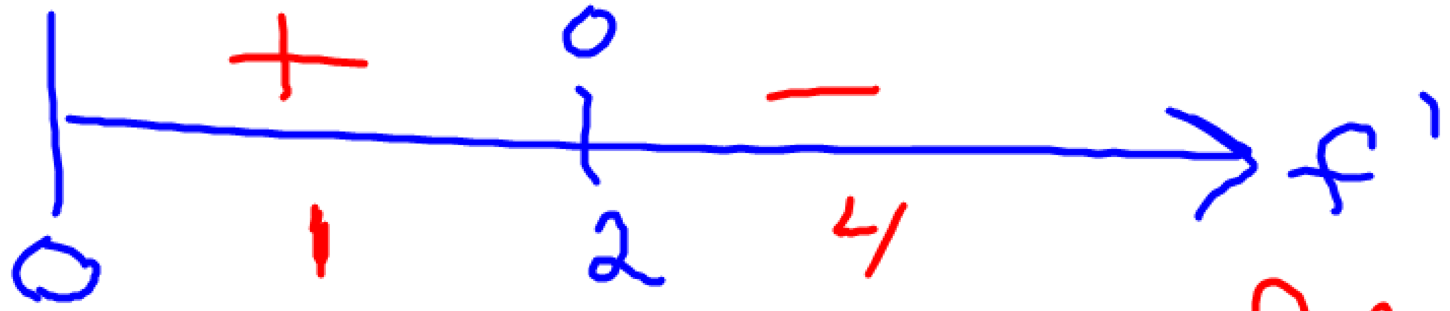
$$6 = 3x$$

$$\boxed{2 = x}$$

$$\underline{f' \infty}$$

$$2x^{1/2} = 0$$

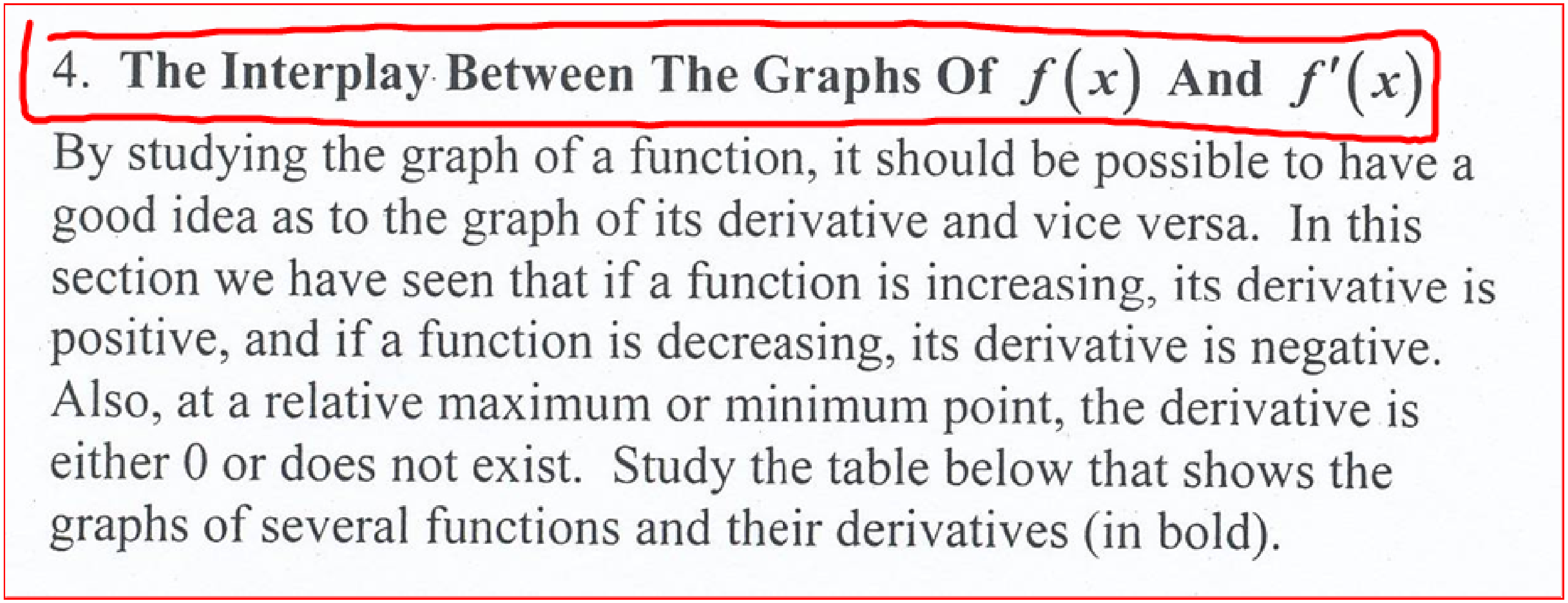
$$\boxed{x = 0}$$



INC
$(0,2)$

Dec
$(2,\infty)$

Rel Max
$f(2) = 4\sqrt{2}$

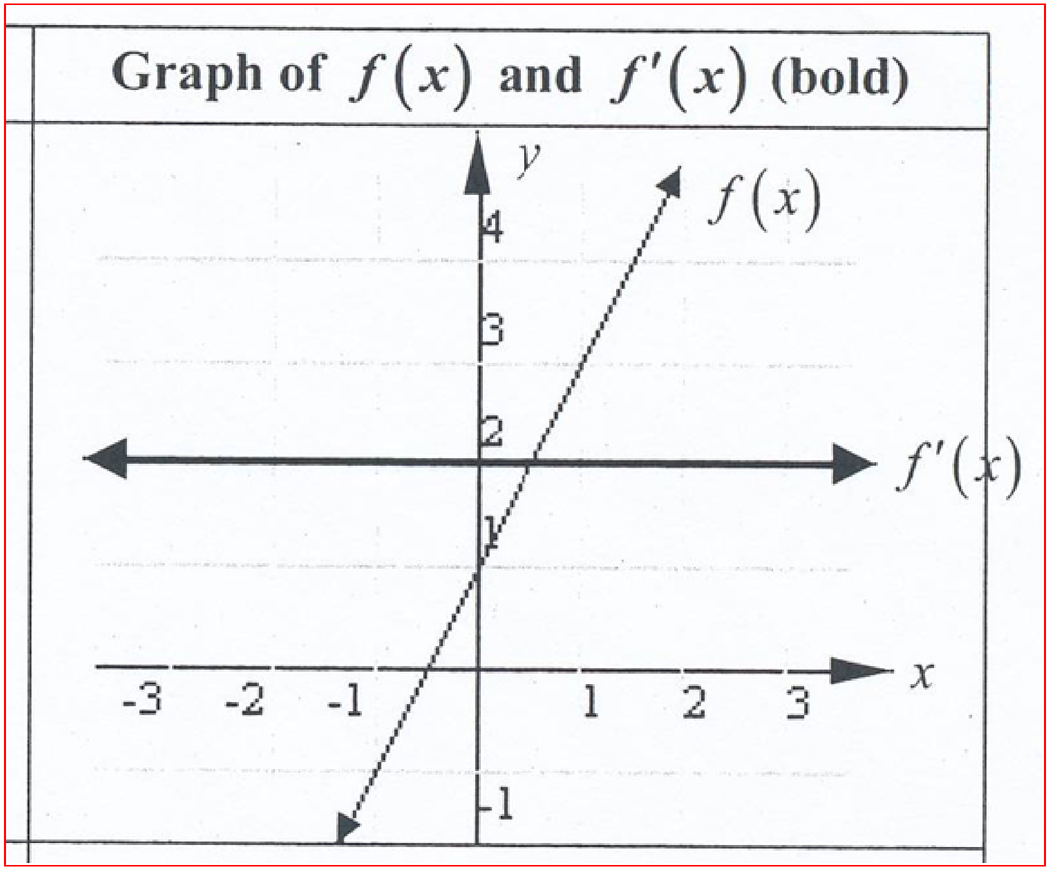


4. The Interplay Between The Graphs Of $f(x)$ And $f'(x)$

By studying the graph of a function, it should be possible to have a good idea as to the graph of its derivative and vice versa. In this section we have seen that if a function is increasing, its derivative is positive, and if a function is decreasing, its derivative is negative. Also, at a relative maximum or minimum point, the derivative is either 0 or does not exist. Study the table below that shows the graphs of several functions and their derivatives (in bold).

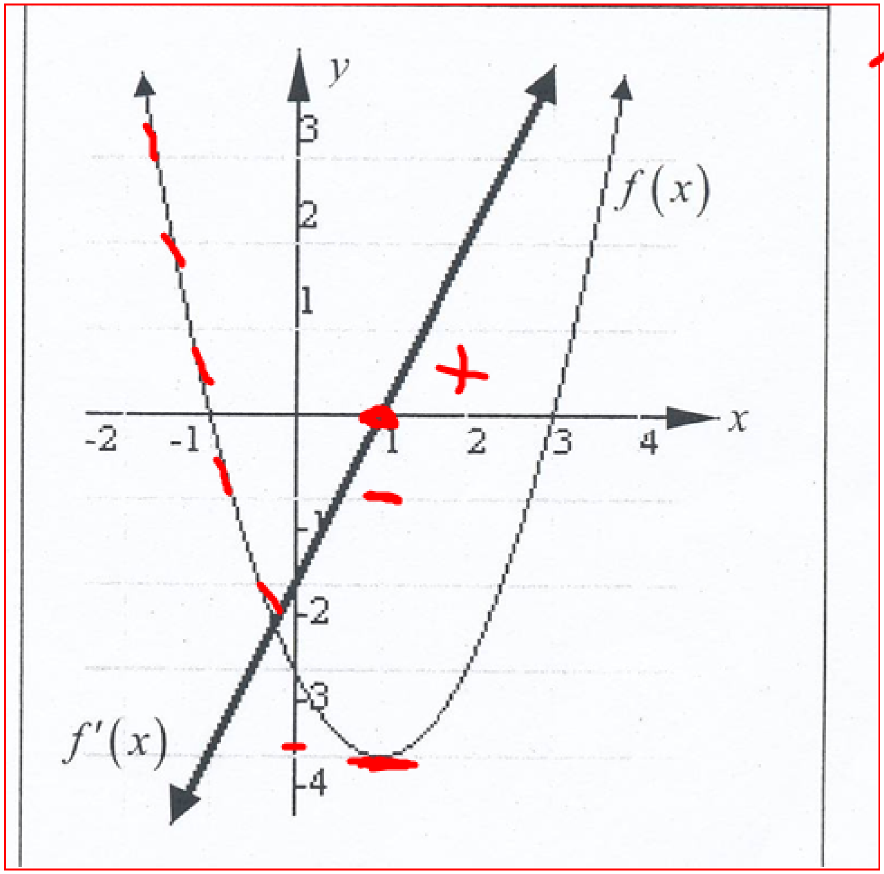
$$f(x)=2x+1$$

$f' = 2$



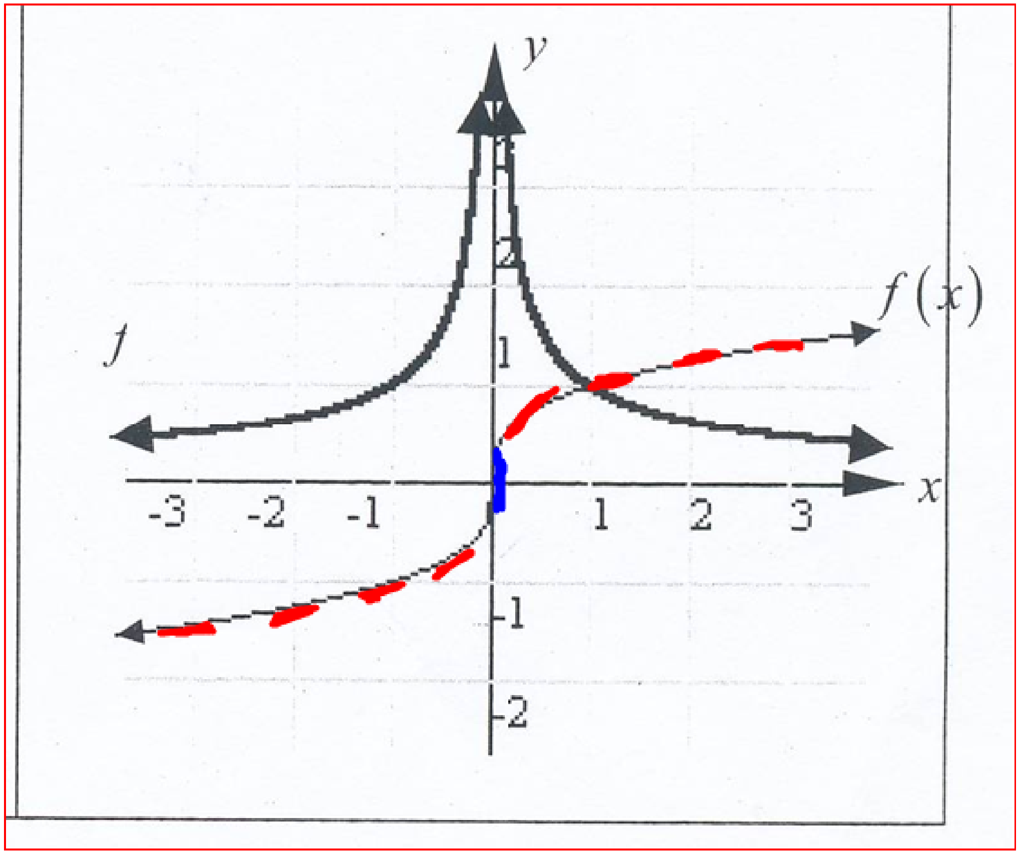
Graph of $f(x)$ and $f'(x)$ (bold)

$$f(x) = x^2 - 2x - 3$$



$f' = 2x - 2$

$$f(x)=\sqrt[3]{x}$$

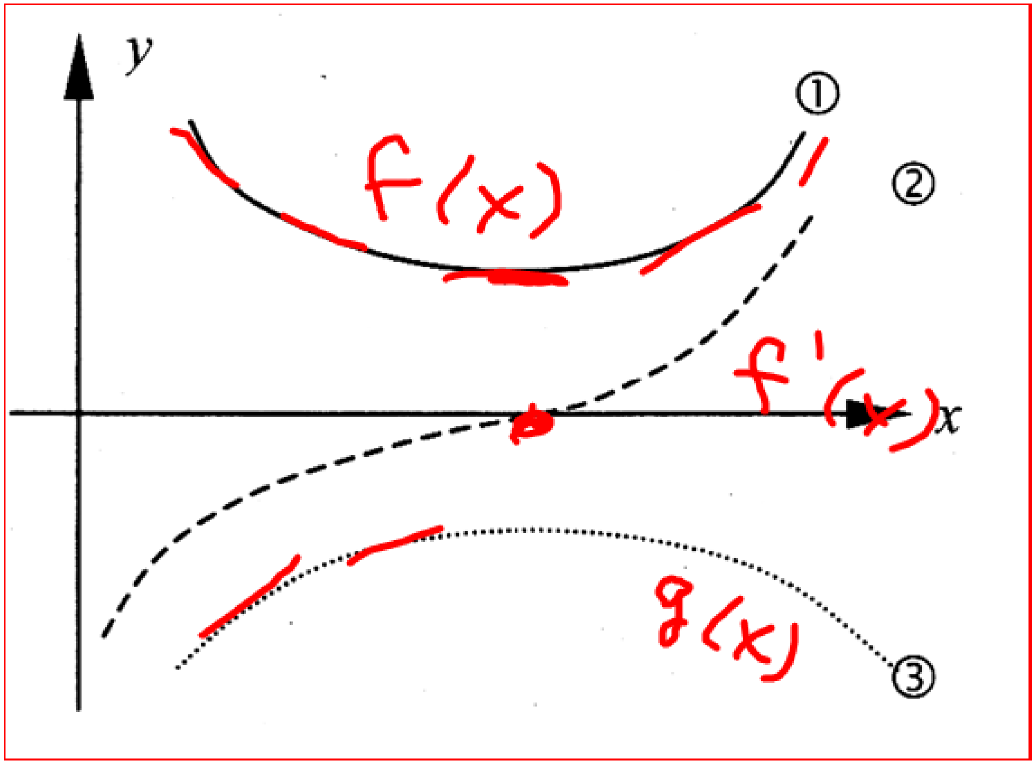


$f' = \frac{1}{3}x^{-2/3}$

$= \frac{1}{3x^{2/3}}$

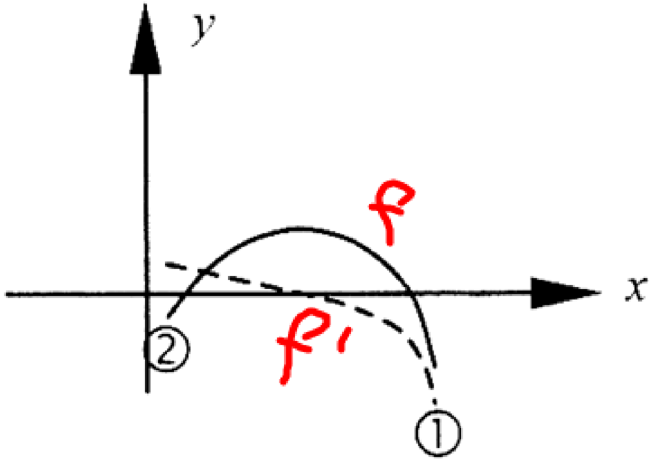
The Big Derivative Puzzle

Shown below are the graphs of $f(x)$, $f'(x)$, and $g(x)$. Which is which?



Your Turn #3

One of the graphs below is that of $f(x)$, the other is that of $f'(x)$. Which is which?



d) $x=5$ f' Δ's from − to +.

e) Max $x=1$ f' always <0 which means f always dec.
Min $x=5$

Example 7 Shown at right is the graph of $f'(x)$ on the interval $[0,6]$. Answer the following questions about $f(x)$. (Justify)

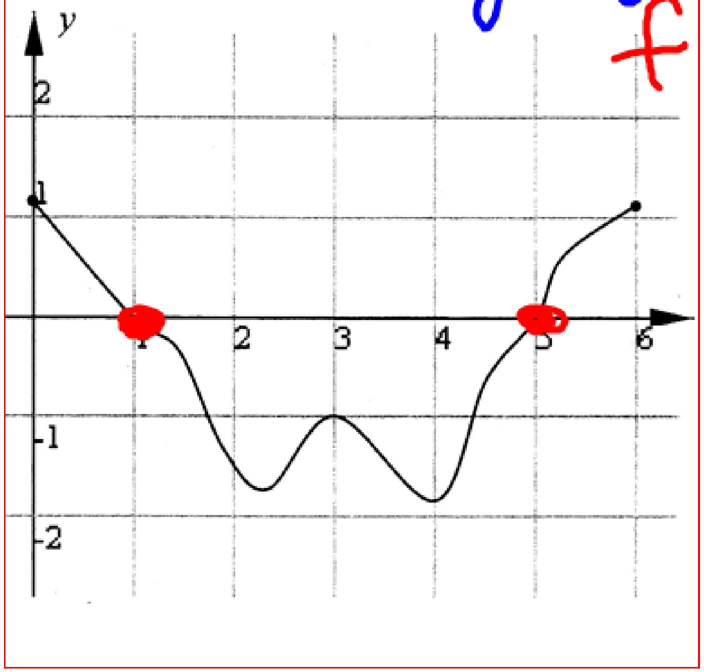
(a) On what open intervals is $f(x)$ increasing?

(b) On what open intervals is $f(x)$ decreasing?

(c) At what x-value is there a relative maximum?

(d) At what x-value is there a relative minimum?

(e) At what x-value on the interval $[1,5]$ does $f(x)$ reach its abs maximum and minimum value?

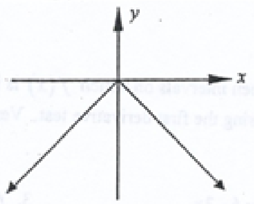
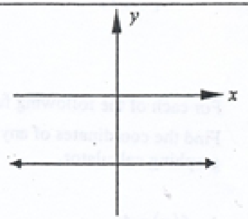
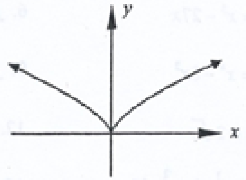
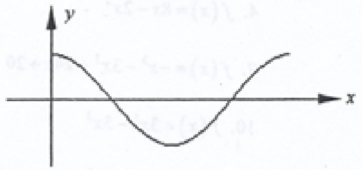
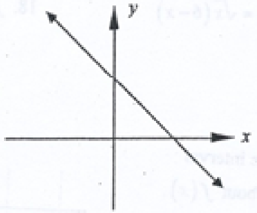
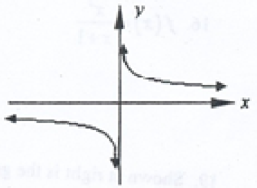
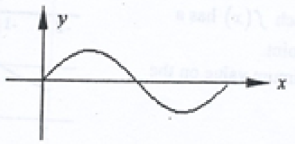
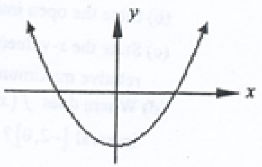
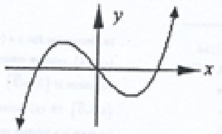
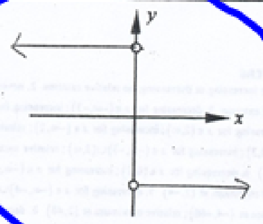


a) $(0,1) \cup (5,6)$ $f'(x)>0$

b) $(1,5)$ $f'(x)<0$

c) $x=1$ f' Δ's from + to −.

20. Match each function with the graph of its derivative. The scales are not necessarily the same from one graph to the next.

$f(x)$	$f'(x)$
(a) 	(i) 
(b) 	(ii) 
(c) 	(iii) 
(d) 	(iv) 
(e) 	(v) 

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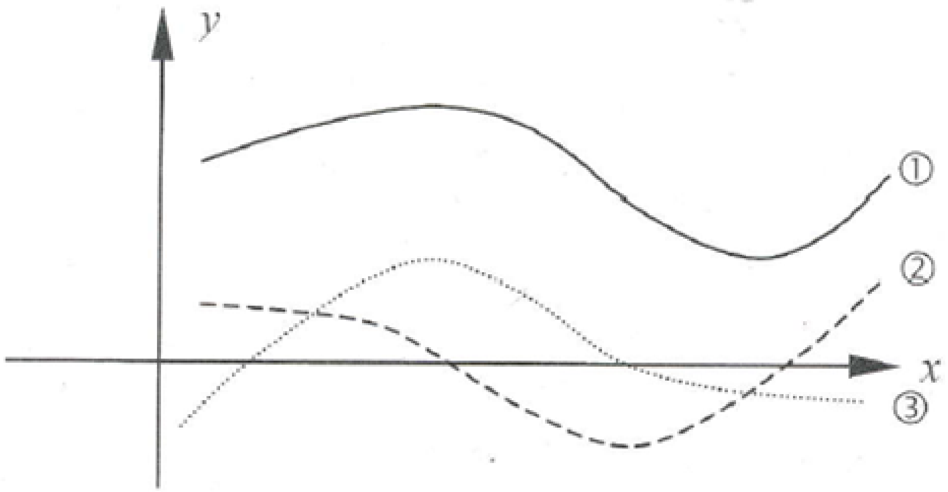
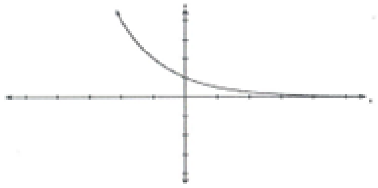
Also Matching Handout and Answering Questions of $f(x)$ from graph of $f'(x)$

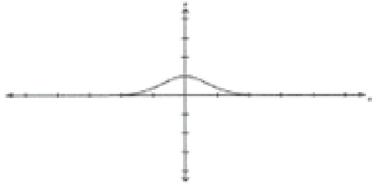
21. Shown below are the graphs of $f(x)$, $f'(x)$, and $g(x)$ (which is not $f'(x)$). Which is which? Explain your reasoning.



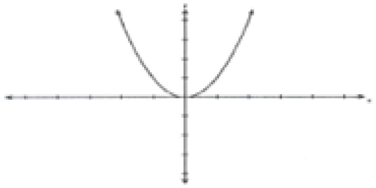
Function Graph G1



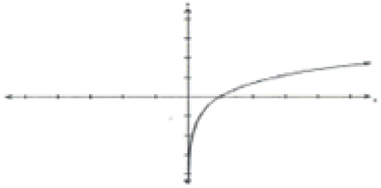
Function Graph G2



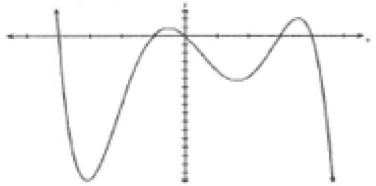
Function Graph G3



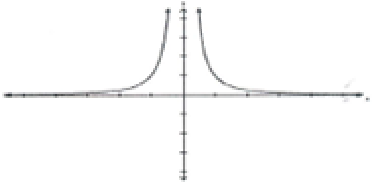
Function Graph G4



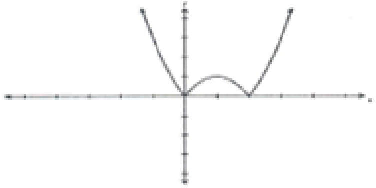
Function Graph G5



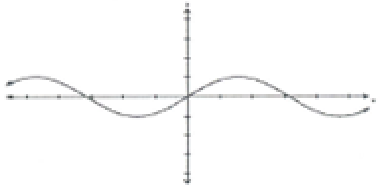
Function Graph G6



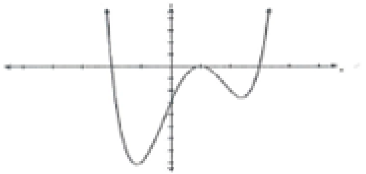
Function Graph G7



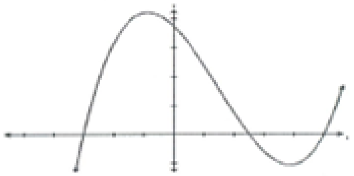
Function Graph G8



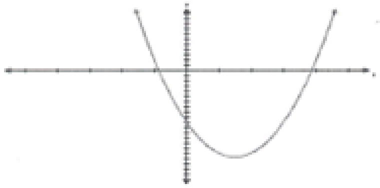
Function Graph G9



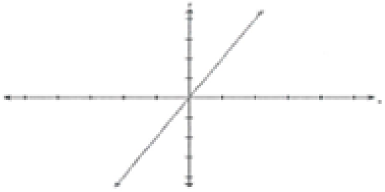
Function Graph G10



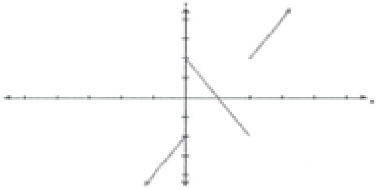
First Derivative Graph dy/dx 1



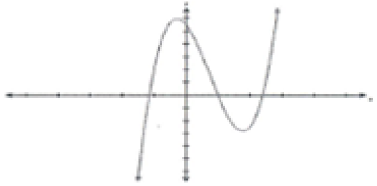
First Derivative Graph dy/dx 2



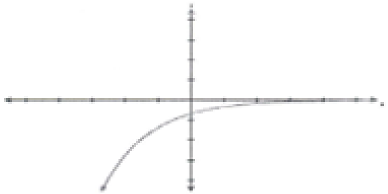
First Derivative Graph dy/dx 3



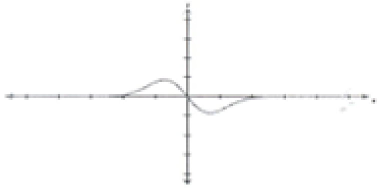
First Derivative Graph dy/dx 4



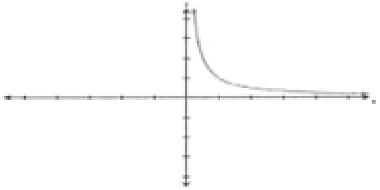
First Derivative Graph dy/dx 5



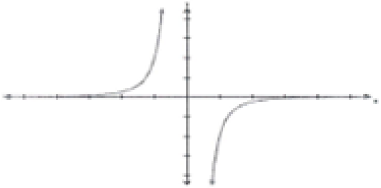
First Derivative Graph dy/dx 6



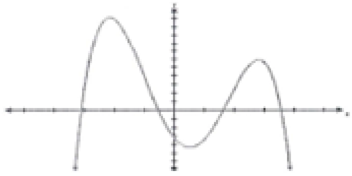
First Derivative Graph dy/dx 7



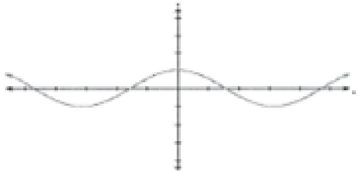
First Derivative Graph dy/dx 8

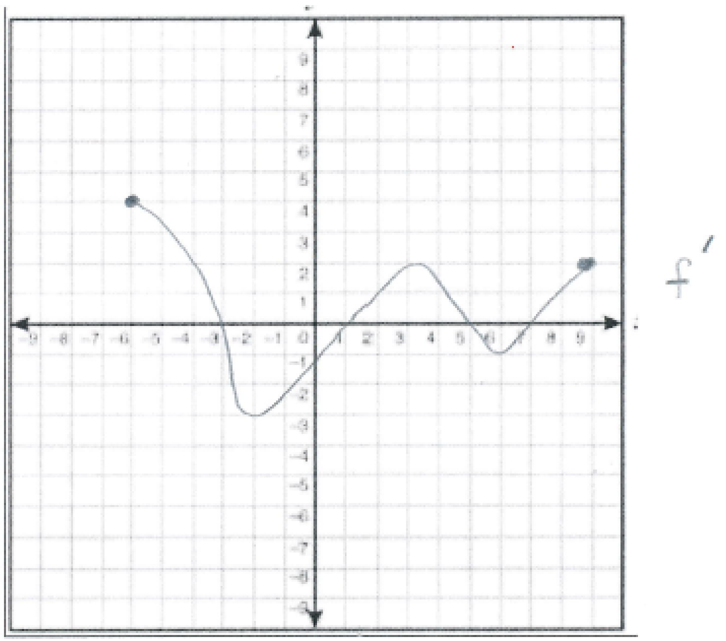


First Derivative Graph dy/dx 9



First Derivative Graph dy/dx 10





Given the **graph of the derivative of $f(x)$** above, answer the following questions in regards to the function $f(x)$.

1. Over interval(s) is $f(x)$ increasing? Justify your answer.

2. Over interval(s) is $f(x)$ decreasing? Justify your answer.

3. List the x-coordinates of any relative maximums. Justify your answer.

4. List the x-coordinates of any relative minimums. Justify your answer.