5.3 Definite Integrals and Antiderivatives

5.3 Definite Integrals and Anti-Derivatives

Learning Targets:

1. SWBAT use the properities of definite integrals to evaluate definite integrals.

2 SWBAT to use definite integral rules to manually evaluate definite integrals.

Properties of Definite Integrals

1. *Order of Integration:* $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$ A definition

2. *Zero:* $\int_a^a f(x)\,dx = 0$ Also a definition

3. *Constant Multiple:* $\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$ Any number k

$\int_a^b -f(x)\,dx = -\int_a^b f(x)\,dx$ $k = -1$

4. *Sum and Difference:* $\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

5. *Additivity:* $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$

a) $\int_0^5 f(x)\,dx + \int_5^7 f(x)\,dx = 10 + 3 = 13$

21. Given $\int_0^5 f(x)\,dx = 10$ and $\int_5^7 f(x)\,dx = 3$, find

(a) $\displaystyle\int_0^7 f(x)\,dx.$ (b) $\displaystyle\int_5^0 f(x)\,dx.$

(c) $\displaystyle\int_5^5 f(x)\,dx.$ $= 0$ (d) $\displaystyle\int_0^5 3f(x)\,dx.$

22. Given $\int_0^3 f(x)\,dx = 4$ and $\int_3^6 f(x)\,dx = -1$, find

(a) $\displaystyle\int_0^6 f(x)\,dx.$ (b) $\displaystyle\int_6^3 f(x)\,dx.$

(c) $\displaystyle\int_3^3 f(x)\,dx.$ (d) $\displaystyle\int_3^6 -5f(x)\,dx.$

23. Given $\int_2^6 f(x)\,dx = 10$ and $\int_2^6 g(x)\,dx = -2$, find

(a) $\displaystyle\int_2^6 [f(x) + g(x)]\,dx.$ (b) $\displaystyle\int_2^6 [g(x) - f(x)]\,dx.$

(c) $\displaystyle\int_2^6 2g(x)\,dx.$ (d) $\displaystyle\int_2^6 3f(x)\,dx.$

24. Given $\int_{-1}^1 f(x)\,dx = 0$ and $\int_0^1 f(x)\,dx = 5$, find

(a) $\displaystyle\int_{-1}^0 f(x)\,dx.$ (b) $\displaystyle\int_0^1 f(x)\,dx - \int_{-1}^0 f(x)\,dx.$

(c) $\displaystyle\int_{-1}^1 3f(x)\,dx.$ (d) $\displaystyle\int_0^1 3f(x)\,dx.$

Example

Let f be a continuous function such that $\int_8^0 f(x)\,dx = -13$ and $\int_0^2 f(x)\,dx = 8$. What is the value of $\int_2^8 (f(x)+4)\,dx$?

$$\int_2^8 f(x)\,dx + \int_2^8 4\,dx$$

$$\int_2^0 f(x)\,dx + \int_0^8 f(x)\,dx + \int_2^8 4\,dx$$

$$-8 + 13 + 4(8-2) = 29$$

CALCULUS AB

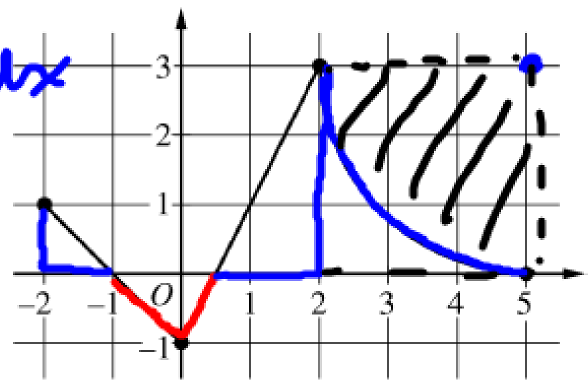
SECTION II, Part B

Time—1 hour

Number of questions—4

NO CALCULATOR IS ALLOWED FOR THESE QUESTIONS.

$\int_{-2}^{5} f(x)\,dx$



$= +\frac{1}{2}(1)(1) - \frac{1}{2}\left(\frac{3}{2}\right)(1) + \frac{1}{2}(3/2)(3) + \left(9 - \frac{\pi}{4}(3)^2\right)$

Graph of f

$= \frac{1}{2} - \frac{3}{4} + \frac{9}{4} + 9 - 9\pi/4$

$= (11 - 9\pi/4)$

3. The continuous function f is defined on the closed interval $-6 \le x \le 5$. The figure above shows a portion of the graph of f, consisting of two line segments and a quarter of a circle centered at the point $(5, 3)$. It is known that the point $(3, 3 - \sqrt{5})$ is on the graph of f.

(a) If $\int_{-6}^{5} f(x)\,dx = 7$, find the value of $\int_{-6}^{-2} f(x)\,dx$. Show the work that leads to your answer.

$$\int_{-6}^{5} f(x)\,dx + \int_{5}^{-2} f(x)\,dx$$

$$= 7 + -\left(11 - \frac{9\pi}{4}\right)$$

$$= 7 - 11 + \frac{9\pi}{4}$$

$$= \frac{9\pi}{4} - 4$$

Basic Integration Rules

$$\int c\,dx = cx$$

$$\int x^n dx = \frac{x^{n+1}}{n+1}, n \neq -1$$

$$\int \cos x\,dx = \sin x$$

$$\int \sin x\,dx = -\cos x$$

$$\int \frac{1}{x} dx = \ln|x|$$

$$\int e^x dx = e^x$$

$$\int e^{kx} dx = \frac{e^{kx}}{k}$$

$$\int a^{kx} dx = \frac{a^{kx}}{k \ln a}$$

Ex.1 Evaluate the following definite integrals:

a) $\int_{3}^{5} 2x\,dx$

$$= x^2 \Big|_3^5 = (5)^2 - (3)^2 = 16$$

$$\text{b) } \int_1^4 (2x+3)\,dx$$

$$= \left(x^2 + 3x\right)\Big|_1^4$$

$$= \left((4)^2 + 3(4)\right) - \left((1)^2 + 3(1)\right)$$

$$= 28 - 4$$

$$= 24$$

c) $\displaystyle\int_{\pi/3}^{\pi/6} \cos x\,dx$

$$= \sin x \Big|_{\pi/3}^{\pi/6}$$

$$= \sin \pi/6 - \sin \pi/3$$

$$= \frac{1}{2} - \frac{\sqrt{3}}{2} = \frac{1-\sqrt{3}}{2}$$

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d) $\displaystyle\int_{\pi/30}^{\pi/10} \sin 5x\,dx$

$$= \frac{-\cos 5x}{5}\Bigg|_{\pi/30}^{\pi/10}$$

$$= -\frac{1}{5}\left(\cos 5\left(\pi/10\right) - \cos 5\left(\pi/30\right)\right)$$

$$= -\frac{1}{5}\left(\cos \pi/2 - \cos \pi/6\right)$$

$$-\frac{1}{5}\left(0 - \sqrt{3}/2\right) = \sqrt{3}/10$$

$$e)\ \int_{\ln 4}^{\ln 6} e^x\,dx$$

$$= e^x \Big|_{\ln 4}^{\ln 6}$$

$$= e^{\ln 6} - e^{\ln 4}$$

$$= 6 - 4$$

$$= 2$$

$$\log_2 2^4$$

$$e^{\log_e 6}$$

$$\text{f) } \int_5^{10} \frac{1}{s}\, ds$$

$$= \ln|s| \Big|_5^{10}$$

$$= \ln 10 - \ln 5$$

$$= \ln\left(\frac{10}{5}\right) = \ln 2$$

g) $\int_1^3 \frac{w^2+1}{w}\,dw$

$$\int_1^3 \left(\frac{w^2}{w} + \frac{1}{w}\right) dw$$

$$\int_1^3 \left(w + \frac{1}{w}\right) dw$$

$$= \frac{w^2}{2} + \ln|w| \Big|_1^3$$

$$= \left(\frac{3^2}{2} + \ln 3\right) - \left(\frac{1^2}{2} + \ln 1\right)$$

$$= \frac{9}{2} + \ln 3 - \frac{1}{2} - \ln 1$$

$$= 4 + \ln 3 - \ln 1$$

$$= 4 + \ln\left(\frac{3}{1}\right)$$

$$= 4 + \ln 3$$

$\ln 1 = 0$

$\ln e = 1$

2003 MC Question

2. $\int_0^1 e^{-4x}\,dx =$

(A) $\frac{-e^{-4}}{4}$ (B) $-4e^{-4}$ (C) $e^{-4}-1$ (D) $\frac{1}{4}-\frac{e^{-4}}{4}$ (E) $4-4e^{-4}$

$$= \left.\frac{e^{-4x}}{-4}\right|_0^1$$

$$= -\frac{1}{4}\left(e^{-4(1)} - e^{-4(0)}\right)$$

$$= -\frac{1}{4}\left(e^{-4} - 1\right) = -\frac{1}{4}e^{-4} + \frac{1}{4}$$

http://archives.math.utk.edu/visual.calculus/4/ftc.10/index.html

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2005 SCORING GUIDELINES

Question 2

The tide removes sand from Sandy Point Beach at a rate modeled by the function R, given by

$$R(t) = 2 + 5\sin\left(\frac{4\pi t}{25}\right).$$

A pumping station adds sand to the beach at a rate modeled by the function S, given by

$$S(t) = \frac{15t}{1 + 3t}.$$

Both $R(t)$ and $S(t)$ have units of cubic yards per hour and t is measured in hours for $0 \le t \le 6$. At time $t = 0$, the beach contains 2500 cubic yards of sand.

(a) How much sand will the tide remove from the beach during this 6-hour period? Indicate units of measure.

(b) Write an expression for $Y(t)$, the total number of cubic yards of sand on the beach at time t.

(c) Find the rate at which the total amount of sand on the beach is changing at time $t = 4$.

(d) For $0 \le t \le 6$, at what time t is the amount of sand on the beach a minimum? What is the minimum value? Justify your answers.

a) $\int_0^6 R(t)\,dt = 31.816 \text{ yd}^3$

b)

$$y(t) = 2500 + \int_0^t \left(S(x) - R(x)\right) dx$$

AP® CALCULUS AB
2005 SCORING GUIDELINES (Form B)

Question 2

A water tank at Camp Newton holds 1200 gallons of water at time $t = 0$. During the time interval $0 \le t \le 18$ hours, water is pumped into the tank at the rate

$$W(t) = 95\sqrt{t}\sin^2\left(\frac{t}{6}\right) \text{ gallons per hour.}$$

During the same time interval, water is removed from the tank at the rate

$$R(t) = 275\sin^2\left(\frac{t}{3}\right) \text{ gallons per hour.}$$

(a) Is the amount of water in the tank increasing at time $t = 15$? Why or why not?

(b) To the nearest whole number, how many gallons of water are in the tank at time $t = 18$?

(c) At what time t, for $0 \le t \le 18$, is the amount of water in the tank at an absolute minimum? Show the work that leads to your conclusion.

(d) For $t > 18$, no water is pumped into the tank, but water continues to be removed at the rate $R(t)$ until the tank becomes empty. Let k be the time at which the tank becomes empty. Write, but do not solve, an equation involving an integral expression that can be used to find the value of k.

a) $W(15) - R(15) = -121.090 < 0$

$\therefore$ amount of H_2O is decreasing at $t = 15$.

b) $A(t) = 1200 + \int_0^{18} \left(W(t) - R(t)\right) dt$

$= 1309.79$

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2004 SCORING GUIDELINES (Form B)

Question 2

For $0 \le t \le 31$, the rate of change of the number of mosquitoes on Tropical Island at time t days is modeled by $R(t) = 5\sqrt{t}\cos\left(\frac{t}{5}\right)$ mosquitoes per day. There are 1000 mosquitoes on Tropical Island at time $t = 0$.

(a) Show that the number of mosquitoes is increasing at time $t = 6$.

(b) At time $t = 6$, is the number of mosquitoes increasing at an increasing rate, or is the number of mosquitoes increasing at a decreasing rate? Give a reason for your answer.

(c) According to the model, how many mosquitoes will be on the island at time $t = 31$? Round your answer to the nearest whole number.

(d) To the nearest whole number, what is the maximum number of mosquitoes for $0 \le t \le 31$? Show the analysis that leads to your conclusion.

a) $R(6) = 4.438 > 0 \therefore$ increasing at $t = 6$

b) $R'(6) = -1.913 < 0$

$\therefore$ at $t = 6$ inc at a dec rate

c) $1000 + \int_0^{31} R(t)\,dt = 964$

d)

t	$A(t)$
0	1000
7.85	1039.357
31	964

$1000 + \int_0^{A} R(t)\,dt$

Assignment

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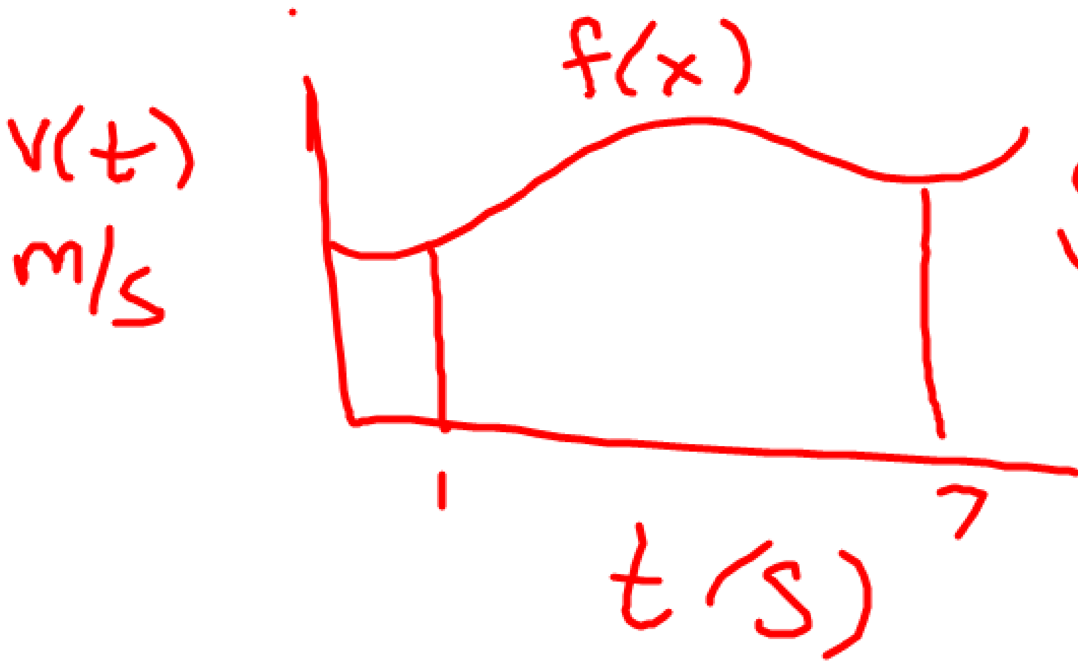
#'s 1-12, 15

Calc 30 Text Page 370

#'s 1-20, 39

Average Value of a Function

Average Value Video



$$\text{ave value} = \frac{1}{7-1\,(s)} \int_1^7 f(x)\,dx \quad m/s$$

Average Value of a Function on an Interval

If *f* is a continuous function on [a,b], then the Average Value of *f* on [a,b] is:

$$= \frac{1}{b-a}\int_a^b f(x)\,dx$$

Example 1

Find the average value of $y = \sin x$ between $x = 0$ and $x = \pi$.

a b

$$\text{ave value} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

$$= \frac{1}{\pi - 0}\int_0^{\pi} (\sin x)\,dx = \frac{2}{\pi}$$

$$= \frac{1}{\pi}\left(-\cos x \Big|_0^{\pi}\right) = -\frac{1}{\pi}\left[\cos\pi - \cos 0\right] = -\frac{1}{\pi}\left[-1-1\right]$$

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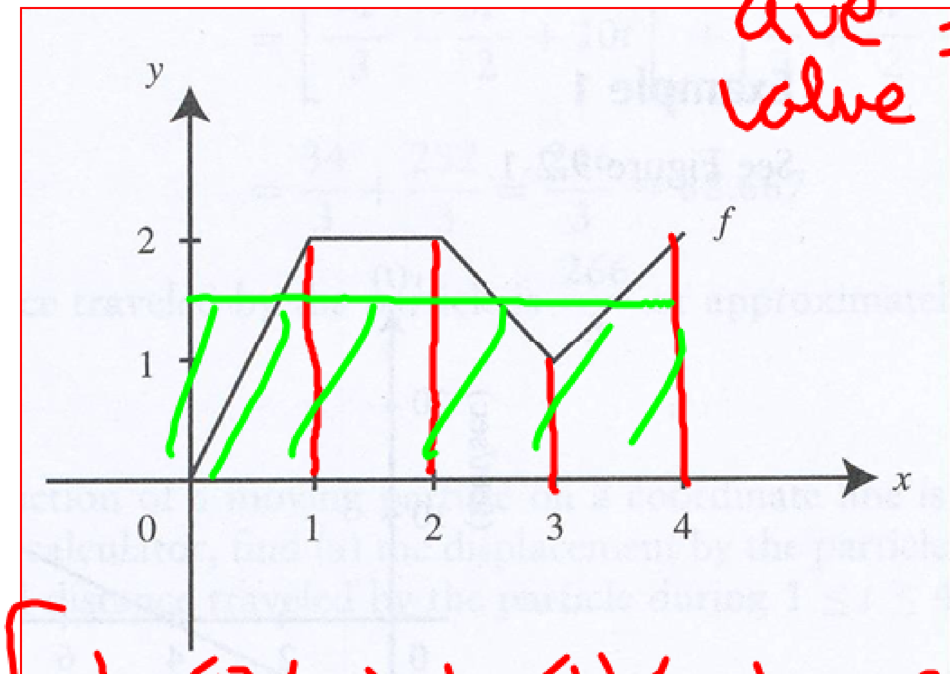
Example 2

The graph of f is shown, find the average value of f on $[0,4]$.

$$\text{ave value} = \frac{1}{4-0}\int_0^4 f(x)\,dx$$



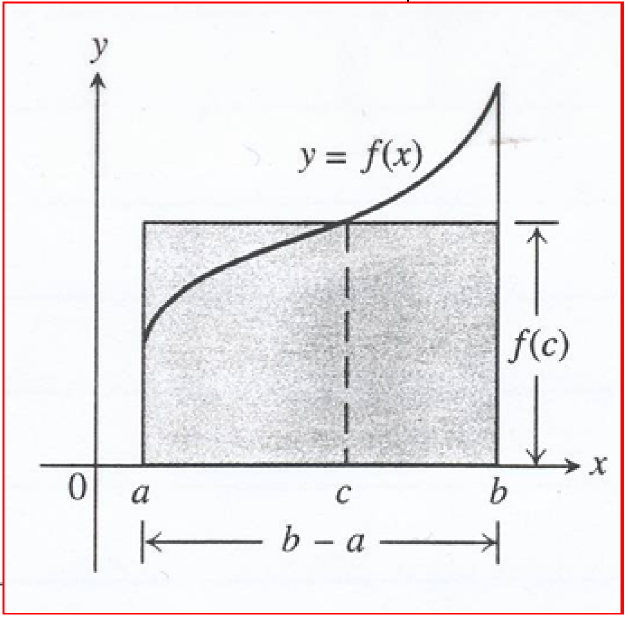
$$= \frac{1}{4}\left[\frac{1}{2}(1)(2) + (1)(2) + 2\left(\frac{1}{2}(1)(1+2)\right)\right]$$

$$= \frac{1}{4}\left[1+2+3\right] = \frac{3}{2}$$

Mean Value Theorem for Integrals

If *f* is a continuous function on [a,b], then at some point *c* in [a,b]:

$$f(c)(b-a)=\int_a^b f(x)\,dx$$



Mean Value Theorem

Example 3

Given $f(x)=\sqrt{x-1}$, verify the hypothesis of the Mean Value Theorem for Integrals for f on $[1,10]$ and find the value of c indicated in the theorem.

$$\text{ave value} = \frac{1}{10-1}\int_1^{10} \sqrt{x-1}\,dx \quad \text{Calc.}$$

$$\text{ave value} = \frac{1}{9}(18) = 2$$

$$(2)^2 = \left(\sqrt{x-1}\right)^2$$

$$4 = x-1$$

$$x = 5$$

Example: Given that the velocity function for a given object by $v(t) = 3t^2 - 18t + 24$, where *t* is given in seconds and *v(t)* is in m/s. Find the average velocity between t=1s and t=3s.

$$\text{ave vel} = \frac{1}{3-1}\int_1^3 (3t^2 - 18t + 24)\,dt$$

$$= \frac{1}{2}\left[t^3 - 9t^2 + 24t \Big|_1^3\right]$$

$$= \frac{1}{2}\left[\left((3)^3 - 9(3)^2 + 24(3)\right) - \left(1 - 9 + 24\right)\right]$$

$$= \frac{1}{2}\left[(27 - 81 + 72) - (16)\right] = \frac{1}{2}(2) = 1\ \text{m/s}$$

CALCULUS AB

SECTION II, Part B

Time—60 minutes

Number of problems—4

No calculator is allowed for these problems.

t (minutes)	0	1	2	3	4	5	6
$C(t)$ (ounces)	0	5.3	8.8	11.2	12.8	13.8	14.5

3. Hot water is dripping through a coffeemaker, filling a large cup with coffee. The amount of coffee in the cup at time t, $0 \le t \le 6$, is given by a differentiable function C, where t is measured in minutes. Selected values of $C(t)$, measured in ounces, are given in the table above.

(a) Use the data in the table to approximate $C'(3.5)$. Show the computations that lead to your answer, and indicate units of measure.

(b) Is there a time t, $2 \le t \le 4$, at which $C'(t) = 2$? Justify your answer.

(c) Use a midpoint sum with three subintervals of equal length indicated by the data in the table to approximate the value of $\frac{1}{6}\int_0^6 C(t)\,dt$. Using correct units, explain the meaning of $\frac{1}{6}\int_0^6 C(t)\,dt$ in the context of the problem.

(d) The amount of coffee in the cup, in ounces, is modeled by $B(t) = 16 - 16e^{-0.4t}$. Using this model, find the rate at which the amount of coffee in the cup is changing when $t = 5$.

a) $c'(3.5) = \dfrac{c(4) - c(3)}{4 - 3}$ $\dfrac{oz}{min}$

$$= \frac{12.8 - 11.2}{1} = 1.6 \ oz/min$$

b) $c'(t) = \dfrac{c(4) - c(2)}{4 - 2}$

$$c'(t) = \frac{12.8 - 8.8}{2} = 2$$

since function is closed, diff.

since average rate Δ between

$2 \le t \le 4$ is 2; the MVT

guarantees $c'(t) = 2$. and

$2 \le t \le 4$.

$$\frac{1}{6}\int_0^6 C(t)\,dt$$

$$= \underset{\text{min}}{\frac{1}{6}}\left[2(5.3+11.2+13.8)\right]^{\text{oz}\cdot\text{min}}$$

$= 10.1$ oz

average amount of coffee in cup from $t=0$ min to $t=6$ min.

d) $B(t) = 16 - 16e^{-.4t}$

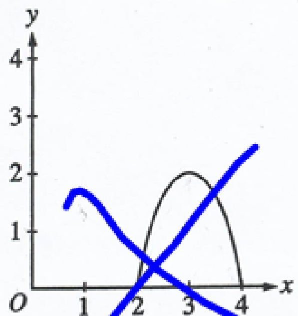
$B'(t) = -16e^{-.4t} \cdot (-.4)$

$B'(t) = 6.4e^{-.4t}$

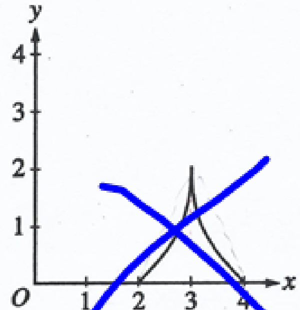
$B'(5) = 6.4e^{-.4(5)}$ oz/min

88. On the closed interval [2, 4], which of the following could be the graph of a function f with the property that $\frac{1}{4-2}\int_2^4 f(t)\,dt = 1$?

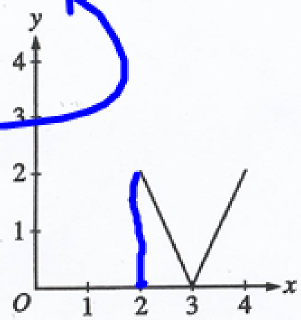
(A)



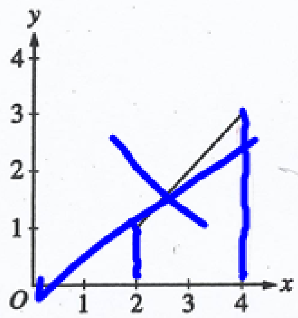
(B)



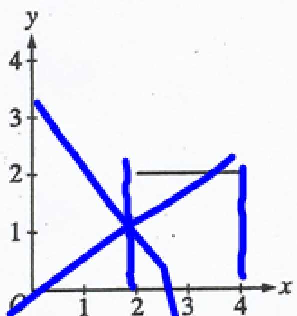
(C)



(D)



(E)



$\int_2^4 f(t)\,dt = 2$

$\int_2^4 f(t)\,dt = 2$

$\frac{1}{2}(1)(2)$

$\frac{1}{2}(2)(1+3)$

Things Not Said By Calculus Students

Assignment

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#'s 21-24 as a group

#'s 25-30

21a) $y = x^2 - 6x + 8 \quad [0, 3]$