5.2 Integration of Rational Functions By Partial Fractions

Expressing a rational function $\left(\frac{x+5}{x^2+x-2}\right)$ as a sum of simpler fractions, called partial fractions, that can be integrated.

$$\int \frac{x+5}{x^2+x-2}\,dx = \int\left(\frac{2}{x-1} - \frac{1}{x+2}\right)dx = 2ln|x-1| - ln|x+2| + C$$

How would we change $\frac{x+5}{x^2+x-2}$ into a partial fraction?

$$\boxed{= \frac{-1}{x+2} + \frac{2}{x-1}}$$

$(x+2)(x-1)$

$$\frac{x+5}{\cancel{(x+2)(x-1)}}\cancel{(x+2)(x-1)} = \frac{A}{\cancel{x+2}} + \frac{B}{\cancel{x-1}}(x+2)\cancel{(x-1)}$$

$$x+5 = A(x-1) + B(x+2)$$

let $x=1$

let $x=-2$

$6 = B(3)$

$2 = B$

$3 = -3A$

$-1 = A$

Things to consider when trying to express a function as a sum of simpler fractions:

$$f(x) = \frac{P(a)}{Q(a)}$$

1. If the degree of the numerator $P(a)$ is less than the degree of the denominator $Q(a)$ then the function can be expressed as a sum of fractions.

2. If the degree of $P(a) \geq Q(a)$ then divide $Q(a)$into $P(a)$ by long division until the remainder is obtained such that degree of $P(a)$ <degree of $Q(a)$

$$f(x) = \frac{P(a)}{Q(a)} = S(x) + \frac{R(x)}{Q(x)}$$

3. Expressing as a Sum of Partial Fractions:

Case 1: Denominators $Q(x)$ is a product of distinct linear factors.

$$\frac{3x+5x}{x(x-3)(x+4)}=\frac{A}{x}+\frac{B}{x-3}+\frac{C}{x+4}$$

Case 2: $Q(x)$ is a product of linear factors, some of which are repeated. ($Q(x)$must have a sum of the repeated factors starting at degree 1 to the degree the factor repeats too.)

$$\frac{x^3 - x + 1}{x^2(x-1)^3} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{(x-1)} + \frac{D}{(x-1)^2} + \frac{E}{(x-1)^3}$$

Case 3: $Q(x)$ has quadratic factors that cannot be reduced.

$$\frac{x}{(x-2)(x^2+1)(x^2+4)} = \frac{A}{(x-2)} + \frac{Bx+C}{(x^2+1)} + \frac{Dx+E}{(x^2+4)}$$

Case 4: $Q(x)$ has repeated quadratic factors.

$$\frac{x^3+x^2+1}{x(x-1)(x^2+x+1)(x^2+1)^3} = \frac{A}{x} + \frac{B}{(x-1)} + \frac{Cx+D}{(x^2+x+1)} + \frac{Ex+F}{(x^2+1)} + \frac{Gx+H}{(x^2+1)^2} + \frac{Ix+J}{(x^2+1)^3}$$

Rewrite as a sum of fractions

$$\frac{1}{x^2 - x - 12}$$

$$\frac{1}{(x-4)(x+3)} = \frac{1/7}{x-4} + \frac{-1/7}{x+3}$$

$$1 = A(x+3) + B(x-4)$$

let $x = -3$

$1 = B(-7)$

$-1/7 = B$

let $x = 4$

$1 = A(7)$

$1/7 = A$

Rewrite as a sum of fractions

$$\frac{6}{x^3 - 5x^2 + 6x}$$

$$\frac{6}{x(x-2)(x-3)} = \frac{\cancel{A}\,1}{x} + \frac{\cancel{B}\,-3}{x-2} + \frac{\cancel{C}\,2}{x-3}$$

$$6 = A(x-2)(x-3) + B(x)(x-3) + C(x)(x-2)$$

let $x = 2$

$6 = B(2)(2-3)$

$-3 = B$

let $x = 0$

$6 = A(0-2)(0-3)$

$1 = A$

let $x = 3$

$6 = C(3)(3-2)$

$2 = C$

Rewrite as a sum of fractions

$$\boxed{\frac{1}{x^3(x^2+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{Dx+E}{x^2+1}}$$

$$1 = A(x^2)(x^2+1) + B(x)(x^2+1) + C(x^2+1) + (Dx+E)(x^3)$$

Rewrite as a sum of fractions

$$\frac{x^3}{x^2 - 2x + 1} = x + 2 + \frac{3x - 2}{x^2 - 2x + 1}$$

$$
\begin{array}{r l}
 & \quad\; x + 2 \\
x^2 - 2x + 1 & \overline{)\, x^3 + 0x^2 + 0x + 0} \\
 & \; -x^3 + 2x^2 - x \\
\hline
 & \qquad\quad 2x^2 - x + 0 \\
 & \qquad -2x^2 + 4x - 2 \\
\hline
 & \qquad\qquad 3x - 2
\end{array}
$$

$$x+2+\frac{3x-2}{(x-1)^2}$$

$$\int x+2+\frac{A}{x-1}+\frac{B}{(x-1)^2}$$

Rewrite as a sum of fractions

$$\frac{x^2+x+4}{x(x+2)(x^2+2)(x^2+5)^2}$$

$$= \frac{A}{x} + \frac{B}{x+2} + \frac{Cx+D}{x^2+2} + \frac{Ex+F}{x^2+5} + \frac{Gx+H}{(x^2+5)^2}$$

Assignment

Handout

#'s 1-6

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#'s 1-6

$$\frac{x^2+1}{x^2-1}$$

$$\begin{array}{r|l} & \quad\quad 1 \\ \hline x^2-1 & \cancel{x^2}+1 \\ & -\cancel{x^2}+1 \\ \hline & \quad\quad 2 \end{array}$$

$$= 1 + \frac{2}{x^2-1}$$

$$= 1 + \frac{2}{(x-1)(x+1)} = \boxed{1 + \frac{A}{x-1} + \frac{B}{x+1}}$$

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Example: $\int \frac{x+16}{x^2+2x-8}\,dx$

$$\frac{x+16}{(x+4)(x-2)} = \boxed{\frac{A}{x+4} + \frac{B}{x-2}}$$

$$x+16 = A(x-2) + B(x+4)$$

let $x=2$

$18 = B(6)$

$3 = B$

let $x=-4$

$12 = A(-6)$

$-2 = A$

$$\int \left(\frac{-2}{x+4} + \frac{3}{x-2} \right) dx$$

$$= -2\ln|x+4| + 3\ln|x-2| + C$$

Example: $\int \frac{1}{x^2 - x - 12}\, dx$

Example:

$$\int \frac{x^2+3x-4}{x^3-4x^2+4x}\,dx$$

$$\frac{x^2+3x-4}{x(x-2)^2} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{(x-2)^2}$$

$$x^2+3x-4 = A(x-2)^2 + Bx(x-2) + Cx$$

let $x=2$

$4+6-4 = 2C$

$3=C$

let $x=0$

$-4 = A(0-2)^2$

$-1=A$

let $x = 1$

$$0 = -1(1-2)^2 + B(1)(1-2) + 3(1)$$

$$0 = -1 - B + 3$$

$$B = 2$$

u sub not required

$3(x-2)^{-2}$

$$\int \left(\frac{-1}{x} + \frac{2}{x-2} + \frac{3}{(x-2)^2} \right) dx$$

$$= -\ln|x| + 2\ln|x-2| - 3(x-2)^{-1} + C$$

Example: $\int \frac{x^3+x}{x-1}\,dx$

$$x^2 + x + 2$$

$$x-1 \,\big)\, x^3 + 0x^2 + x + 0$$

$$-\ x^3 + x^2$$

$$x^2 + x$$

$$-\ x^2 + x$$

$$2x + 0$$

$$-\ 2x + 2$$

$$2$$

$$\int \left(x^2 + x + 2 + \frac{2}{x-1}\right) dx$$

$$= \frac{x^3}{3} + \frac{x^2}{2} + 2x + 2\ln|x-1| + C$$

Example: $\int \frac{4x^2-4x+6}{x^3-x^2-6x}\,dx$

$$\frac{4x^2-4x+6}{x(x-3)(x+2)} = \frac{A}{x} + \frac{B}{x-3} + \frac{C}{x+2}$$

$$4x^2-4x+6 = A(x-3)(x+2) + Bx(x+2) + Cx(x-3)$$

let $x=3$

$36-12+6 = 15B$

$30 = 15B$

$2 = B$

let $x=0$

$6 = A(-6)$

$-1 = A$

let $x=-2$

$16+8+6 = 10C$

$30 = 10C$

$3 = C$

Example: $\int \frac{4x^2-4x+6}{x^3-x^2-6x}\,dx$

$$\frac{4x^2-4x+6}{x(x-3)(x+2)} = \frac{A}{x} + \frac{B}{x-3} + \frac{C}{x+2}$$

$$4x^2-4x+6 = A(x-3)(x+2) + Bx(x+2) + Cx(x-3)$$

let $x = 3$

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$2 = B$

let $x = 0$

$6 = A(-6)$

$-1 = A$

let $x = -2$

$16+8+6 = 10C$

$30 = 10C$

$3 = C$

$$= \int -\frac{1}{x} + \frac{2}{x-3} + \frac{3}{x+2}$$

$$= -\ln|x| + 2\ln|x-3| + 3\ln|x+2| + C$$

Example:

$$\int \frac{x^2+2x-1}{2x^3+3x^2-2x}\,dx$$

$$\frac{1}{2}\ln|x| + \frac{1}{10}\ln|2x-1| - \frac{1}{10}\ln|x+2| + C$$

$$\int \frac{\left(\frac{1}{2}\right)}{x} + \frac{\frac{1}{5}}{(2x-1)} - \frac{\frac{1}{10}}{x+2}$$

$$= \frac{1}{2}\ln|x| + \frac{1}{10}\ln|2x-1| - \frac{1}{10}\ln|x+2| + C$$

Example: $\int \frac{x+1}{x^2(x+2)}\,dx$

$$\frac{x+1}{x^2(x+2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+2}$$

$$x+1 = A(x)(x+2) + B(x+2) + Cx^2$$

let $x=0$	let $x=-2$	let $x=1$
$1 = 2B$	$-1 = 4C$	$2 = 3A + 3B + C$
$\frac{1}{2} = B$	$-\frac{1}{4} = C$	$A = \frac{1}{4}$

Assignment: Handout