5.2 Definite Integrals

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Learning Targets:

1. SWBAT define a definite integral and list all the terminolgy associated to one.

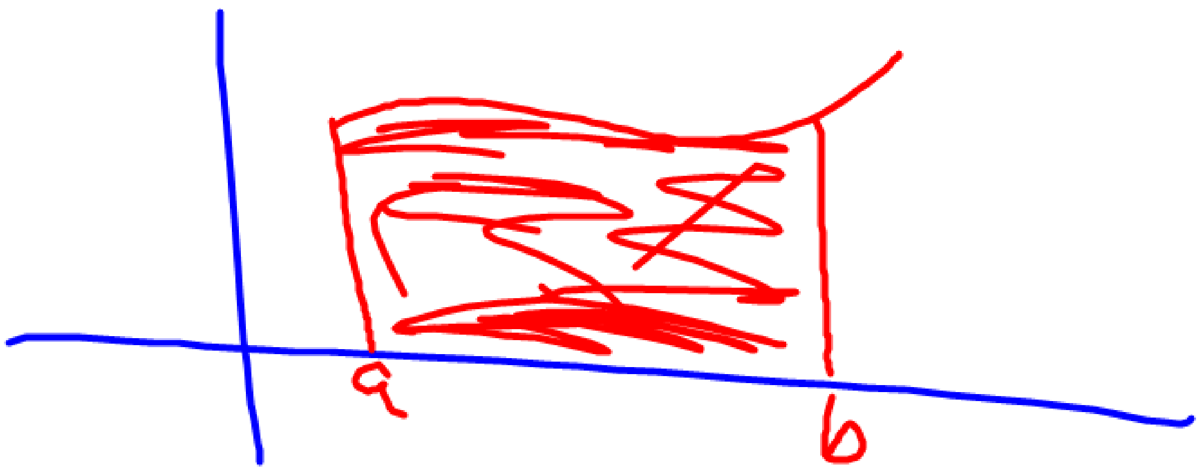
2 SWBAT evaluate definite integrals by find the area under the curve of the function.

3. SWBAT evaluate the definite intergral of a constant function.

4. SWBAT evaluate a definite intergral of function using a graphing calculator.

In 5.1 we calculated LRAM,RRAM and MRAM.
These are actually called Riemann Sums.

We did these informally so now we will discuss a more formal construction that does not limit us to non-negative functions.



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$$\text{EXACT AREA} = \lim_{N \to \infty} \left[\sum_{i=1}^{N} f(c_i)\,\Delta x \right] \qquad [A, B]$$

Step 1: FIND $\Delta x = \dfrac{B - A}{N}$

Step 2: FIND $c_i = A + (\Delta x)\, i$

Step 3: FIND THE GENERAL SOLUTION IN TERMS OF "N" $\sum_{i=1}^{N} f(c_i)\,\Delta x$

Step 4: FIND THE "EXACT" SOLUTION $\lim_{N \to \infty}$

Find the exact area using sigma notation and a Riemann Sum

$f(x) = x^2$ over the interval [0,2]

1) $\Delta x = \frac{b-a}{n} = \frac{2-0}{n} = \frac{2}{n}$

2) $c_i = a + i\Delta x$

$= 0 + i\frac{2}{n}$

$c_i = \frac{2}{n}i$

3) $\sum_{i=1}^{n} f(c_i)\Delta x = \sum_{i=1}^{n}\left(\frac{2}{n}i\right)^2 \cdot \frac{2}{n}$

$$\lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{2i}{n}\right)^2 \frac{2}{n} = \int_0^2 x^2\,dx$$

Your turn

Find the exact area using sigma notation and a Riemann Sum

$f(x) = \sqrt{x}$ over the interval [9,25]

1) $\Delta x = \frac{b-a}{n} = \frac{25-9}{n} = \frac{16}{n}$

2) $c_i = a + i\Delta x$

$c_i = 9 + \frac{16}{n} i$

3) $\sum_{i=1}^{n} f(c_i)\Delta x = \sum_{i=1}^{n} \sqrt{9+\frac{16}{n} i} \cdot \frac{16}{n}$

4) $\lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{9+\frac{16}{n} i} \cdot \frac{16}{n} = \int_{9}^{25} \sqrt{x}\, dx$

Sometimes we may have to convert integrals from sigma notation to integral notation and vice versa.

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(\hat{x}_i)\,\Delta x_i,$$

The interval [-1,3] is partitioned into n sub-intervals of equal length $\Delta x = \frac{4}{n}$.

Let c_k denote the midpoint of the k^{th} subinterval. Express the limit

$\lim_{n\to\infty} \sum_{k=1}^{n} (3(c_k)^2 - 2c_k + 5)\Delta x$ as an integral.

Example

Write the following summation as a definite integral:

$$\sum_{k=1}^{n} \sin\left(5+\frac{3k}{n}\right)\frac{3}{n} = \int_{5}^{8} \sin x \, dx$$

$$\Delta x = \frac{b-a}{n}$$

$$\frac{3}{n} = \frac{b-a}{n}$$

$$\frac{3}{n} = \frac{b-5}{n} \quad (b = 8)$$

30. Which of the following limits is equal to $\int_2^5 x^2\,dx$?

(A) ~~$\lim_{n\to\infty} \sum_{k=1}^{n} \left(2+\frac{k}{n}\right)^2 \frac{1}{n}$~~

(B) $\lim_{n\to\infty} \sum_{k=1}^{n} \left(2+\frac{k}{n}\right)^2 \frac{3}{n}$

(C) ~~$\lim_{n\to\infty} \sum_{k=1}^{n} \left(2+\frac{3k}{n}\right)^2 \frac{1}{n}$~~

(D) $\lim_{n\to\infty} \sum_{k=1}^{n} \left(2+\frac{3k}{n}\right)^2 \frac{3}{n}$

EXACT AREA = $\lim_{N\to\infty} \left[\sum_{i=1}^{N} f(c_i)\,\Delta x\right]$ [A, B]

Step 1: FIND $\Delta x = \frac{B-A}{N}$

Step 2: FIND $c_i = A + (\Delta x)\,i$

Step 3: FIND THE GENERAL SOLUTION $\sum_{i=1}^{N} f(c_i)\,\Delta x$ IN TERMS OF "N"

Step 4: FIND THE "EXACT" SOLUTION $\lim_{N\to\infty}$

$\Delta x = \frac{5-2}{n} = \frac{3}{n}$

$c_i = 2 + \frac{3}{n} i$

Your turn, rewrite the definite integral as limit of a Riemann Sum.

$$\int_0^3 e^x \, dx = ?$$

$$\Delta x = \frac{3-0}{n} = \frac{3}{n}$$

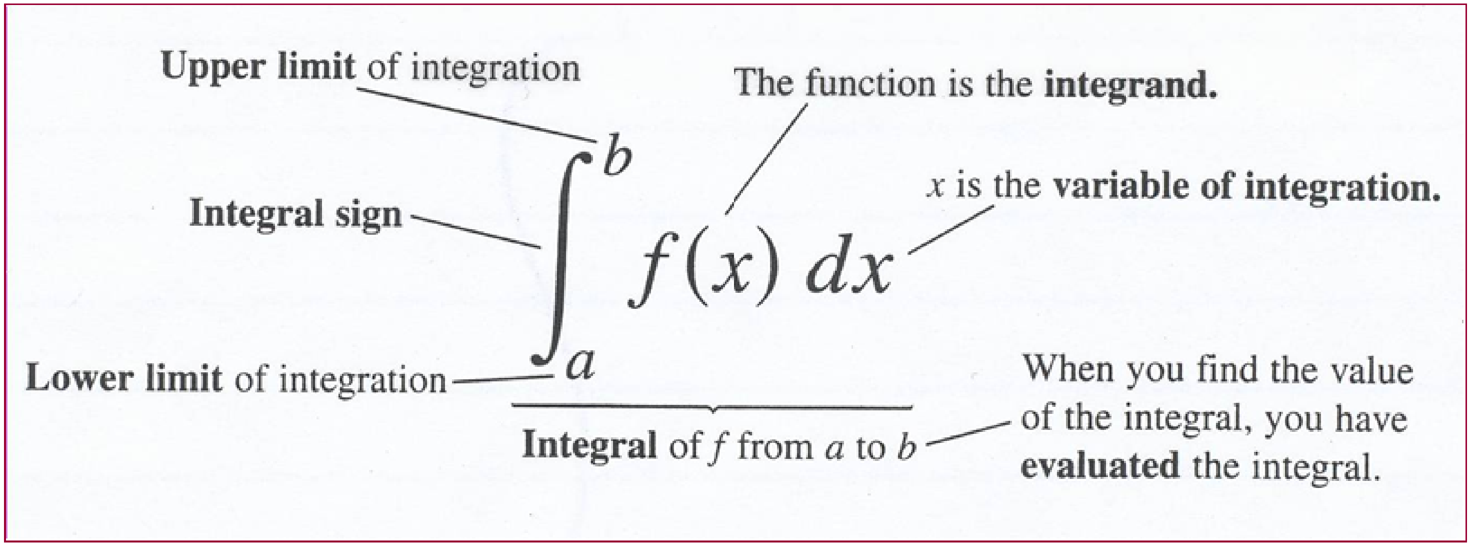
$$c_i = \frac{3}{n} i$$

$$\lim_{n \to \infty} \sum_{i=1}^{n} e^{\frac{3}{n} i} \cdot \frac{3}{n}$$

Your turn again!

$$\int_{\pi}^{2\pi} cosx dx$$

Terminology and Notation of Integration

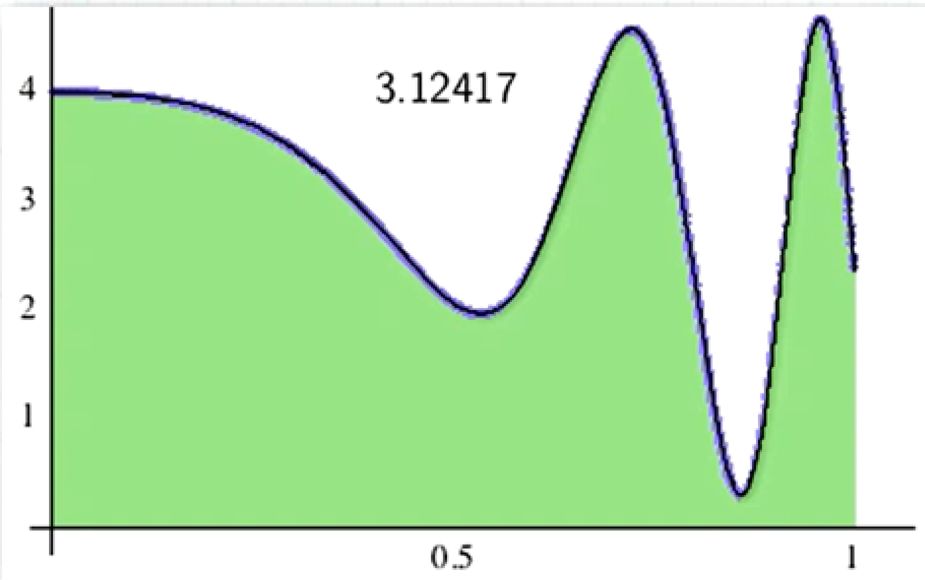


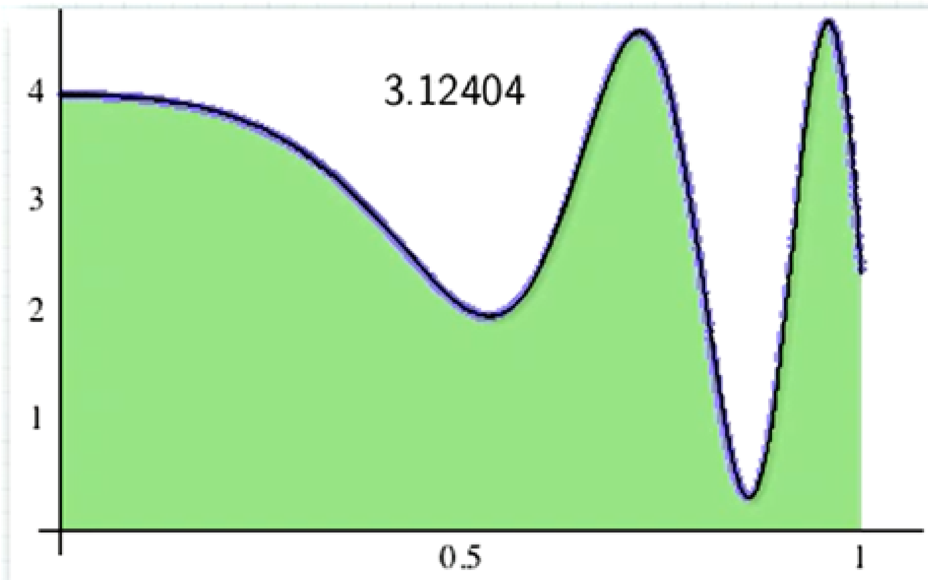
The definite integral really is calculating the net area under a curve over a given interval.

Theorem: The Existence of Definite Integrals

All continuous functions are integrable. That is, if a function is continuous on an interval [a,b], then its definite integral over [a,b] exists.

Integrals and Signed Area

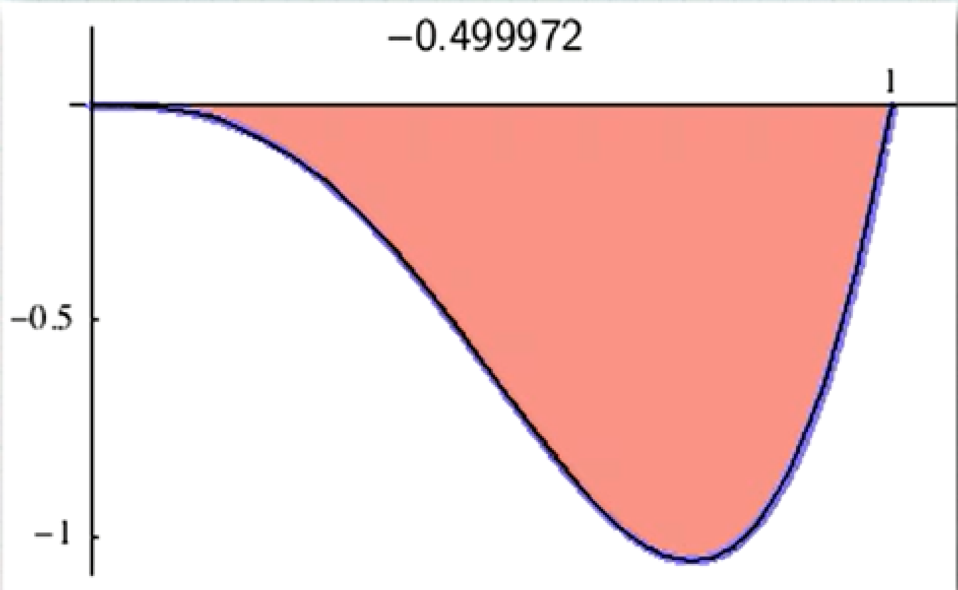


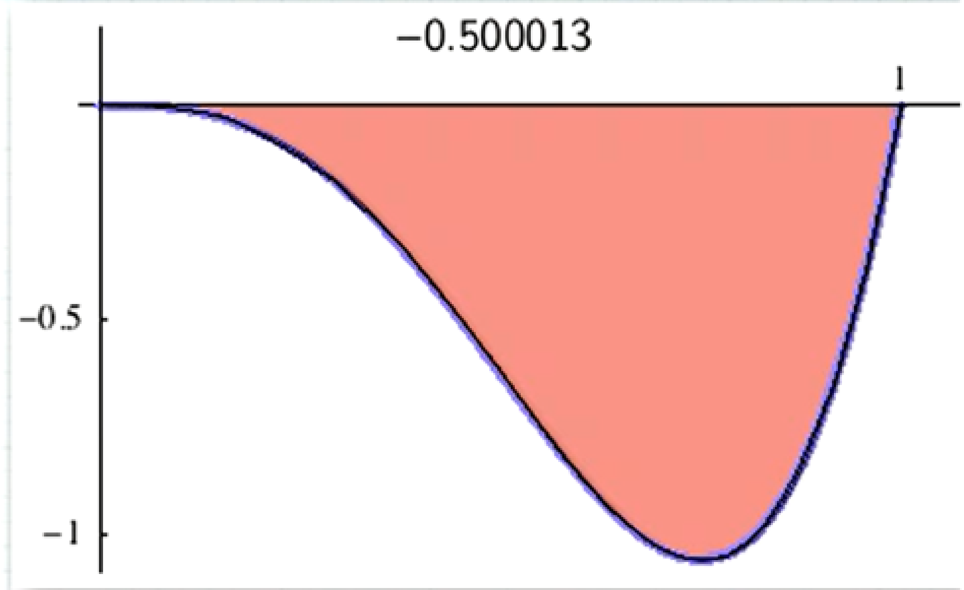


Let f be continuous on $[a, b]$. If $f(x) \geq 0$ for all x in $[a, b]$, and if $f(x) > 0$ for *some* x in $[a, b]$, then

$$\int_a^b f(x)\,dx > 0$$

and equals the area of the region bounded by the graph of f and the x-axis between $x = a$ and $x = b$.

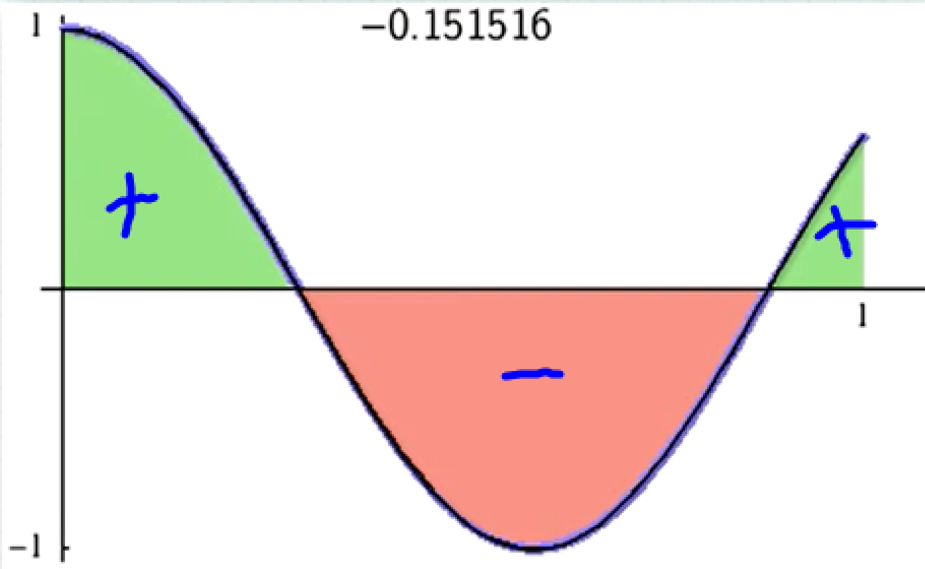


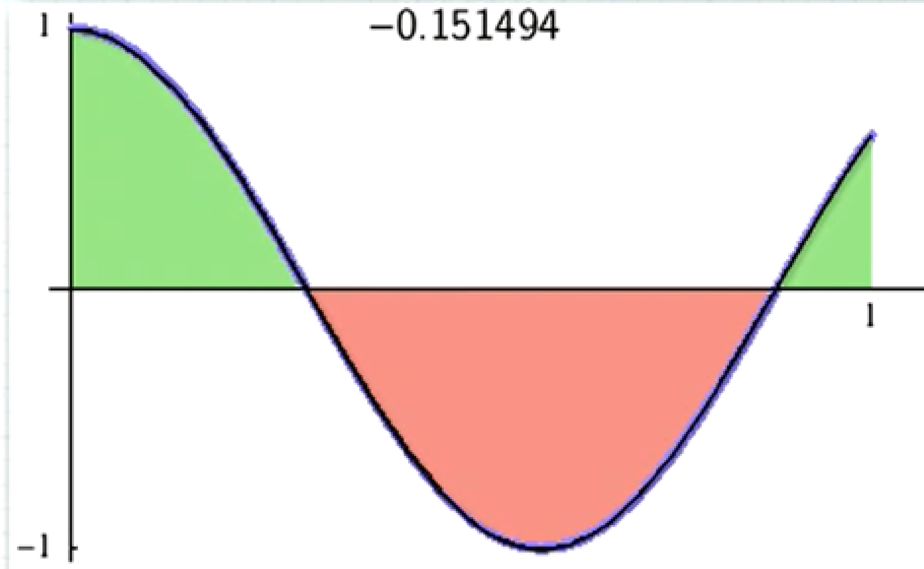


Let f be continuous on $[a,b]$. If $f(x) \leq 0$ for all x in $[a,b]$, and if $f(x) < 0$ for *some* x in $[a,b]$, then

$$\int_a^b f(x)\,dx < 0$$

and $-\int_a^b f(x)\,dx$ equals the area of the region bounded by the graph of f and the x-axis between $x = a$ and $x = b$.



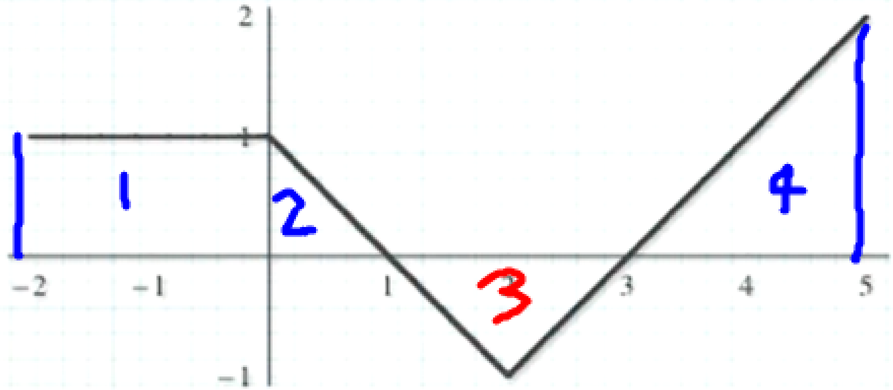


$\int_a^b f(x)\,dx$ equals the *difference* between the area under the graph of f *above* the x-axis and the area above the graph of f *below* the x-axis between $x = a$ and $x = b$.

This is the **signed** (or *net*) **area** of the region bounded by the graph of f and the x-axis between $x = a$ and $x = b$.

Example Let $f(x) = \begin{cases} 1, & \text{if } x < 0 \\ 1 - x, & \text{if } 0 \le x < 2 \\ x - 3, & \text{if } 2 \le x \end{cases}$. Find $\int_{-2}^{5} f(x)\,dx$.

area under curve



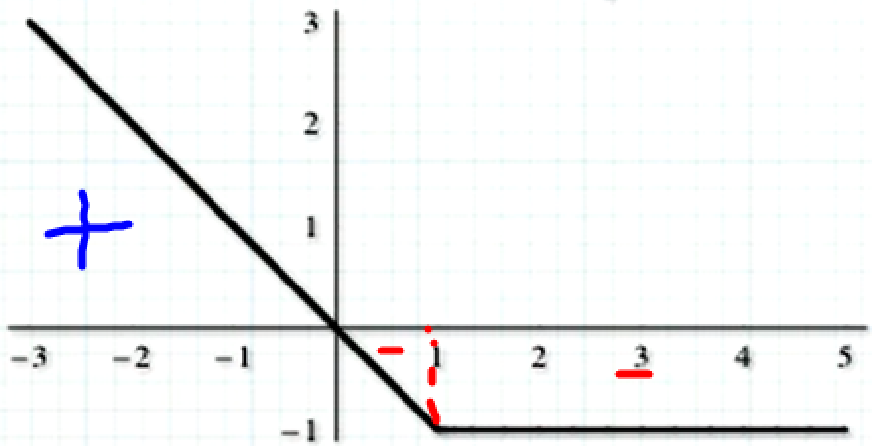
$$(2)(1) + \tfrac{1}{2}(1)(1) - \tfrac{1}{2}(2)(1) + \tfrac{1}{2}(2)(2)$$

$$= 2 + \tfrac{1}{2} - 1 + 2$$

$$= \frac{7}{2}$$

$$\int_{-2}^{5} f(x) = 7/2$$

Example Let $f(x) = \begin{cases} -x, & \text{if } x < 1 \\ -1, & \text{if } 1 \leq x \end{cases}$. Find $\int_{-3}^{5} f(x)\,dx$.



$\frac{1}{2}(3)(3) - \frac{1}{2}(1)(1) - (4)(1)$

$= 0$

Defn: Area Under a Curve

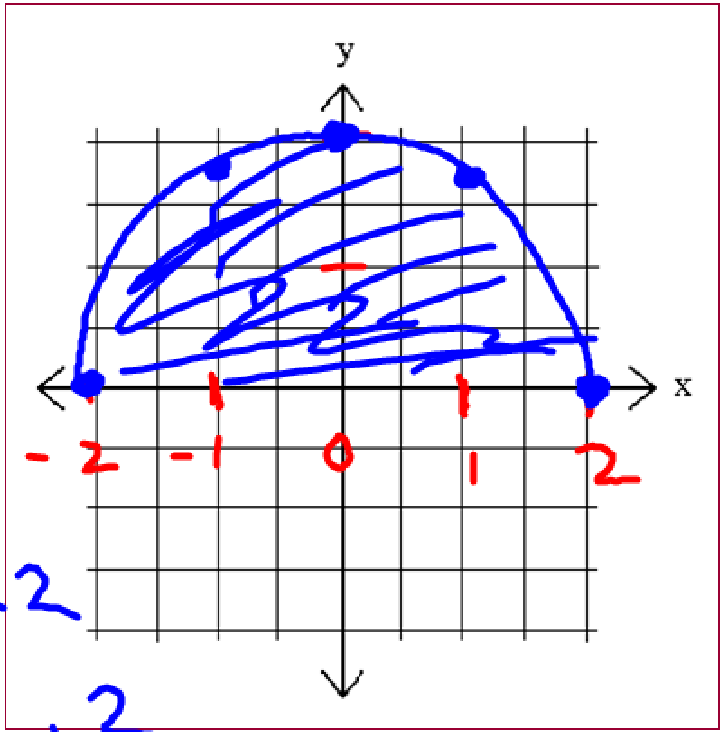
If $y = f(x)$ is integrable over a closed interval $[a,b]$ then the area under the curve $y = f(x)$ from a to b is given by :

$$A = \int_a^b f(x)dx$$

Ex.1 Evaluate the following integrals.

a) $\int_{-2}^{2} \sqrt{4-x^2}\,dx$

x	y
-2	0
-1	$\sqrt{3}$
0	2
1	$\sqrt{3}$
2	0

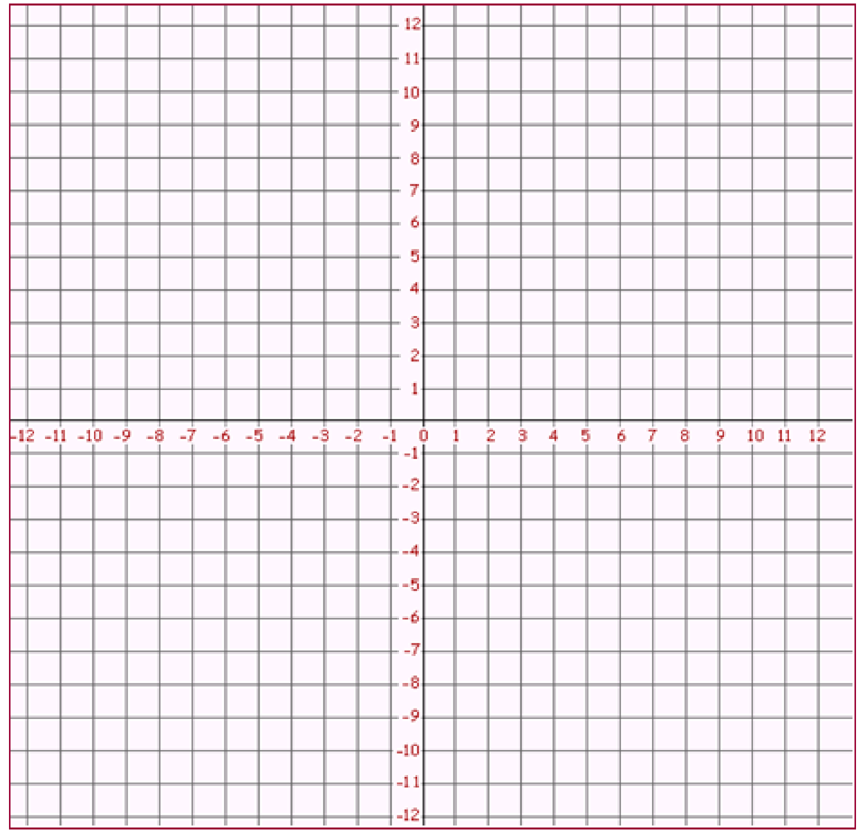


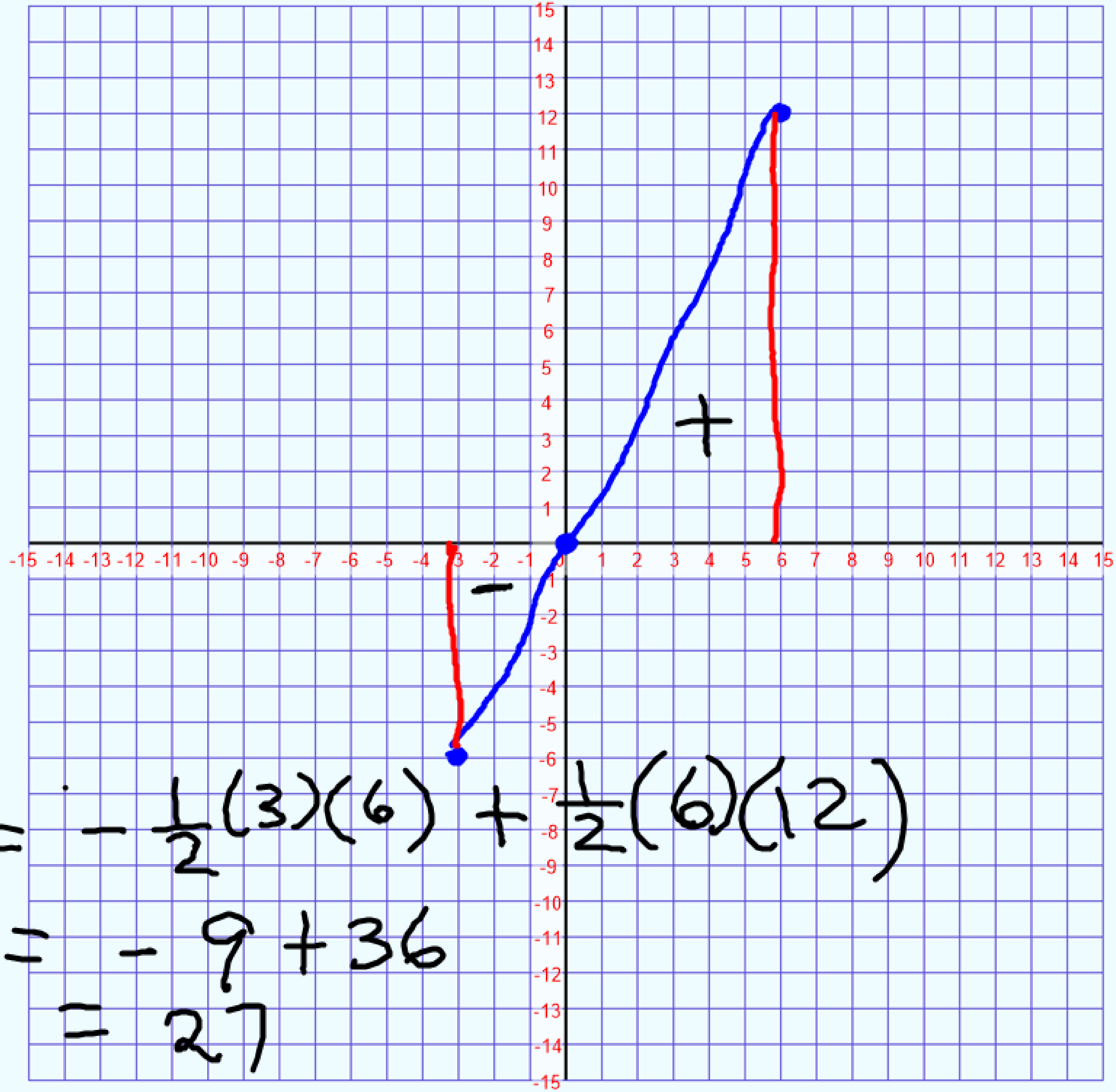
$= \frac{1}{2}\pi r^2$

$= \frac{1}{2}\pi(2)^2$

$= 2\pi$

$$\text{b) } \int_{-3}^{6} 2x\,dx$$





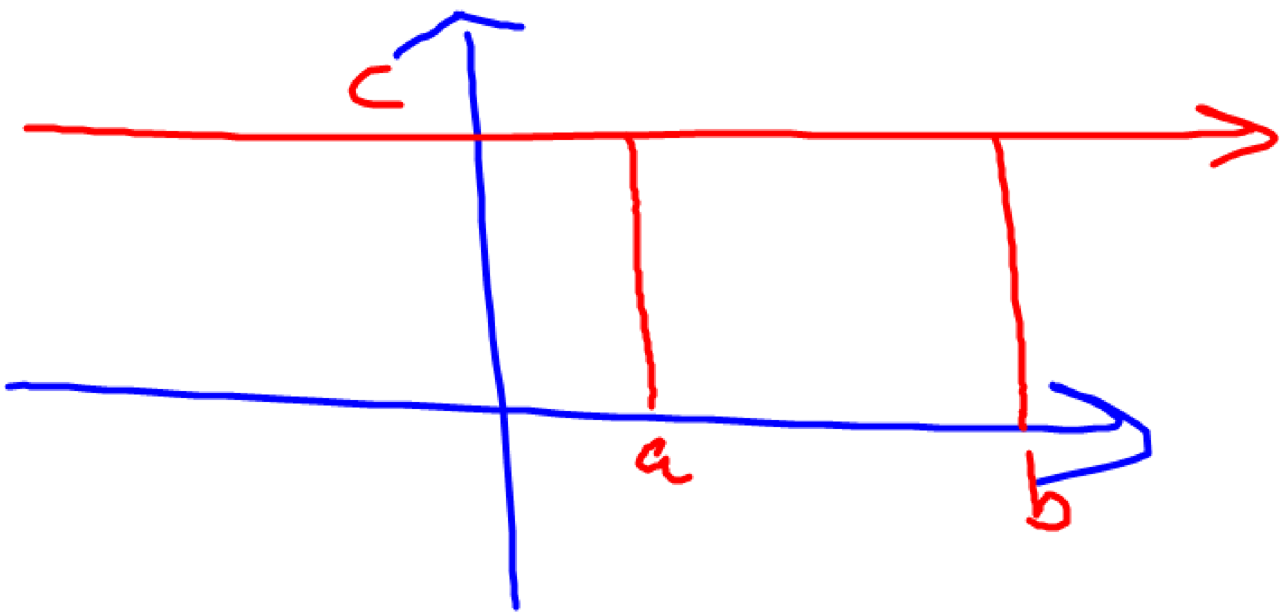
$$= -\frac{1}{2}(3)(6) + \frac{1}{2}(6)(12)$$

$$= -9 + 36$$

$$= 27$$

Integrals of a Constant Function

If $f(x) = c$, where c is a constant on the interval $[a,b]$ then $\int_a^b f(x)dx = \int_a^b c\,dx = c(b-a)$



Integrals on a Calculator

We can use our graphing calculator to also evaluate definite integrals.

Ex.2 Evaluate using a calculator:

a) $\int_{-1}^{2} x \sin x dx$

b) $\int_{0}^{5} e^{-x^2} dx$

c) $\int_{0}^{1} \frac{4}{1+x^2} dx$

$$\int_1^5 (x^2 + 2x)\,dx$$

Write as a Riemann Sum

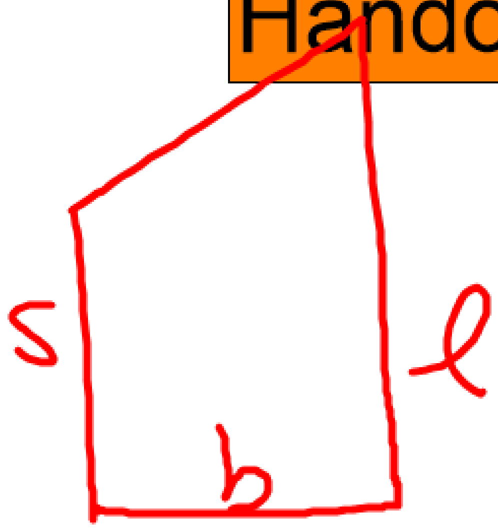
$$\Delta x = \frac{5-1}{n} = \frac{4}{n}$$

$$c_i = 1 + \frac{4}{n} i$$

$$\lim_{n \to \infty} \sum_{i=1}^{n} \left(\left(1 + \frac{4}{n} i\right)^2 + 2\left(1 + \frac{4}{n} i\right) \right) \frac{4}{n}$$

Assignment
AP Text Page 267
#'s 1-12,
13-19(area of a trapezoid),21-25
39,40 (use a calculator for these two)

Handout back of booklet.



$\frac{1}{2}(base)(s+\ell)$