

## 5.2 Definite Integrals

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### Learning Targets:

1. SWBAT define a definite integral and list all the terminology associated to one.
2. SWBAT evaluate definite integrals by find the area under the curve of the function.
3. SWBAT evaluate the definite intergral of a constant function.
4. SWBAT evaluate a definite intergral of function using a graphing calculator.



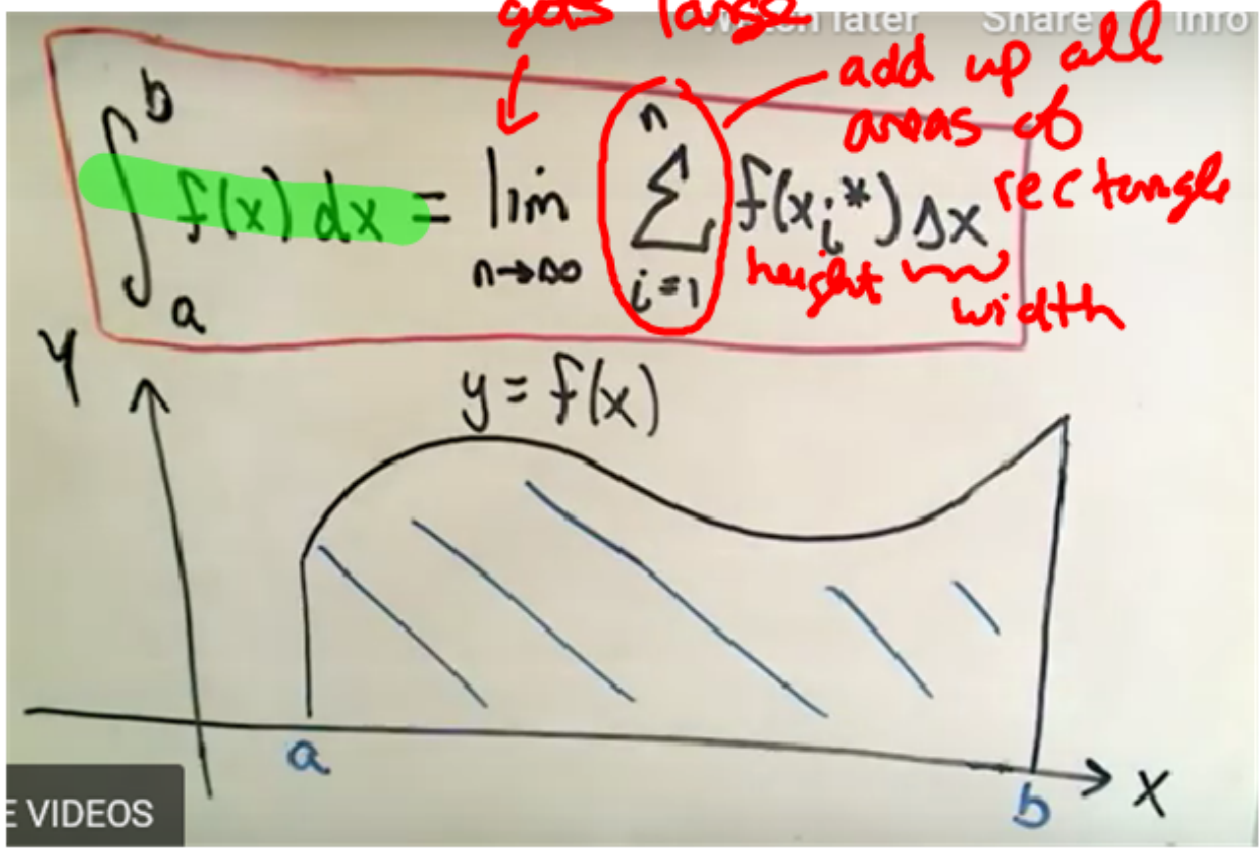
In 5.1 we calculated **LRAM,RRAM** and **MRAM**.  
These are actually called **Riemann Sums**.

We did these informally so now we will discuss a more formal construction that does not limit us to non-negative functions.



## The Definite Integral

# rectangles gets large



$$\text{EXACT AREA} = \lim_{N \rightarrow \infty} \left[ \sum_{i=1}^N f(c_i) \Delta x \right] \quad [A, B]$$

**Step 1:** FIND  $\Delta x = \frac{B - A}{N}$

**Step 2:** FIND  $c_i = A + (\Delta x) i$

**Step 3:** FIND THE GENERAL SOLUTION IN TERMS OF "N"  $\sum_{i=1}^N f(c_i) \Delta x$

**Step 4:** FIND THE "EXACT" SOLUTION  $\lim_{N \rightarrow \infty}$

Find the exact area using sigma notation and a Riemann Sum

$f(x) = x^2$  over the interval  $[0, 2]$

$$\int_0^2 x^2 dx$$

$$\textcircled{1} \Delta x = \frac{b-a}{n} = \frac{2-0}{n} = \frac{2}{n}$$

$$\textcircled{2} c_i = a + \Delta x i$$

$$c_i = 0 + \frac{2i}{n}$$

$$c_i = \frac{2i}{n}$$

$$\textcircled{3} \sum_{i=1}^n f(c_i) \Delta x$$

$$\sum_{i=1}^n \left(\frac{2i}{n}\right)^2 \frac{2}{n}$$

$$\textcircled{4} \lim_{n \rightarrow \infty} \sum_{i=1}^n \left(\frac{2i}{n}\right)^2 \frac{2}{n}$$



Your turn

Find the exact area using sigma notation and a Riemann Sum

$f(x) = \sqrt{x}$  over the interval  $[9, 25]$

$$\int_9^{25} \sqrt{x} \, dx$$

$$\textcircled{1} \quad \Delta x = \frac{b-a}{n} = \frac{25-9}{n} = \frac{16}{n}$$

$$\textcircled{2} \quad c_i = a + \Delta x i \\ = 9 + \frac{16}{n} i$$

$$\textcircled{3} \sum_{i=1}^n \sqrt{9 + \frac{16}{n}i} \cdot \frac{16}{n}$$

$$\textcircled{4} \lim_{n \rightarrow \infty} \sum_{i=1}^n \sqrt{9 + \frac{16}{n}i} \left(\frac{16}{n}\right)$$

Sometimes we may have to convert integrals from sigma notation to integral notation and vice versa.

$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(\hat{x}_i) \Delta x_i,$$

The interval  $[-1, 3]$  is partitioned into  $n$  sub-intervals of equal length  $\Delta x = \frac{4}{n}$ .

Let  $c_k$  denote the midpoint of the  $k^{\text{th}}$  subinterval. Express the limit

$\lim_{n \rightarrow \infty} \sum_{k=1}^n (3(c_k)^2 - 2c_k + 5) \Delta x$  as an integral.

$$\int_{-1}^3 (3x^2 - 2x + 5) dx$$

## Example

Write the following summation as a definite integral:

$$\sum_{k=1}^n \sin\left(5 + \frac{3k}{n}\right) \frac{3}{n}$$

$a + \Delta x i$

$$a = 5$$
$$b = 8$$

$$\int_5^8 \sin x \, dx$$

$$\Delta x = \frac{b-a}{n}$$
$$\frac{3}{n} = \frac{b-5}{n}$$

30. Which of the following limits is equal to  $\int_2^5 x^2 dx$  ?

(A)  $\lim_{n \rightarrow \infty} \sum_{k=1}^n \left(2 + \frac{k}{n}\right)^2 \frac{1}{n}$

(B)  $\lim_{n \rightarrow \infty} \sum_{k=1}^n \left(2 + \frac{k}{n}\right)^2 \frac{3}{n}$

(C)  $\lim_{n \rightarrow \infty} \sum_{k=1}^n \left(2 + \frac{3k}{n}\right)^2 \frac{1}{n}$

(D)  $\lim_{n \rightarrow \infty} \sum_{k=1}^n \left(2 + \frac{3k}{n}\right)^2 \frac{3}{n}$

EXACT AREA =  $\lim_{N \rightarrow \infty} \left[ \sum_{i=1}^N f(c_i) \Delta x \right] \quad [A, B]$

**Step 1:** FIND  $\Delta x = \frac{B - A}{N}$

**Step 2:** FIND  $c_i = A + (\Delta x) i$

**Step 3:** FIND THE GENERAL SOLUTION  $\sum_{i=1}^N f(c_i) \Delta x$   
IN TERMS OF "N"

**Step 4:** FIND THE "EXACT" SOLUTION  $\lim_{N \rightarrow \infty}$

$$\Delta x = \frac{5 - 2}{n} = \frac{3}{n}$$

$$c_k = 2 + \frac{3}{n} k$$

Your turn, rewrite the definite integral as limit of a Riemann Sum.

$$\int_0^3 e^x dx = ?$$

$$\Delta x = \frac{3-0}{n} = \frac{3}{n}$$

$$c_i = 0 + \frac{3}{n}i$$

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n \left( e^{\frac{3}{n}i} \right) \frac{3}{n}$$

Your turn again!

$$\int_{\pi}^{2\pi} \cos x dx$$

$$\Delta x = \frac{2\pi - \pi}{n} = \frac{\pi}{n}$$

$$c_i = \pi + \frac{\pi}{n} i$$

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n \cos\left(\pi + \frac{\pi}{n} i\right) \frac{\pi}{n}$$

Assignment  
Handout  
and  
Matching Game



## Apply Your Understanding of Definite Integrals

In the chart below, a definite integral has been provided for you. If a definite integral has been provided, write the corresponding limit of a Riemann sum.

	Definite Integral	Limit of Riemann Sum
1.	$\int_0^6 \sqrt{2x+1} dx$	
2.	$\int_{-2}^3 x^2 - 3 dx$	
3.	$\int_1^6 3x - 4 dx$	
4.	$\int_{-2}^4 x^3 dx$	

5.

$$\int_{-2}^0 \sqrt{x^2 + 1} dx$$

6.

$$\int_2^6 5x + 7 dx$$

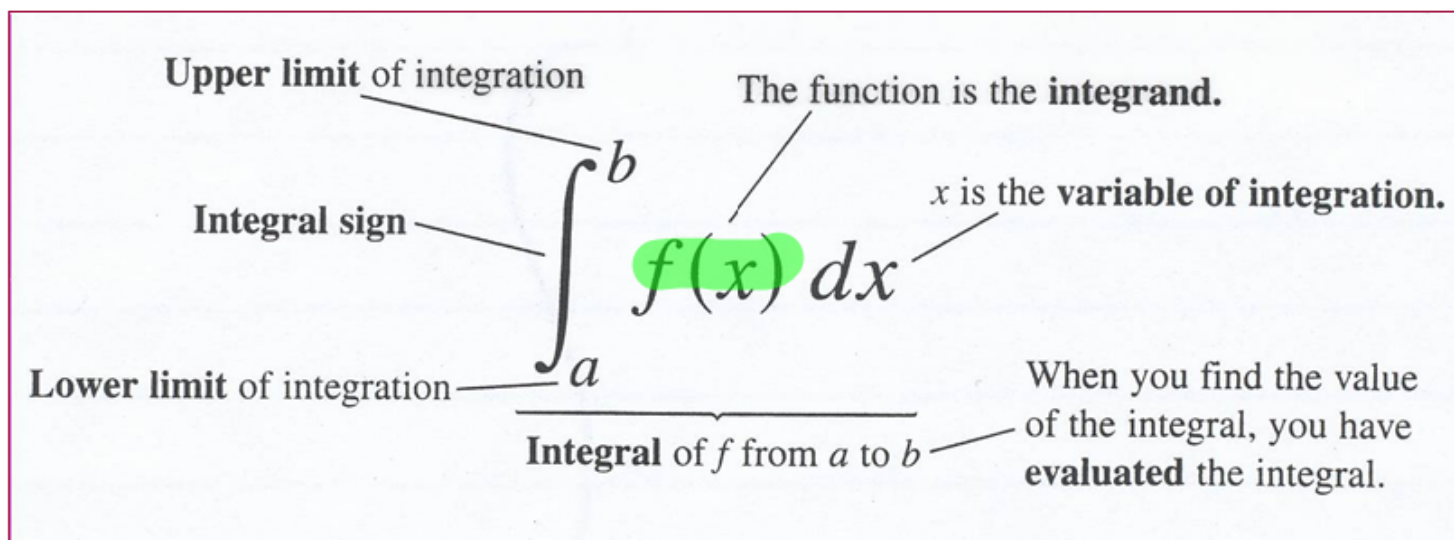
7.

$$\int_0^4 6x^2 - 2 dx$$

8.

$$\int_1^3 4x^3 - 1 dx$$

# Terminology and Notation of Integration

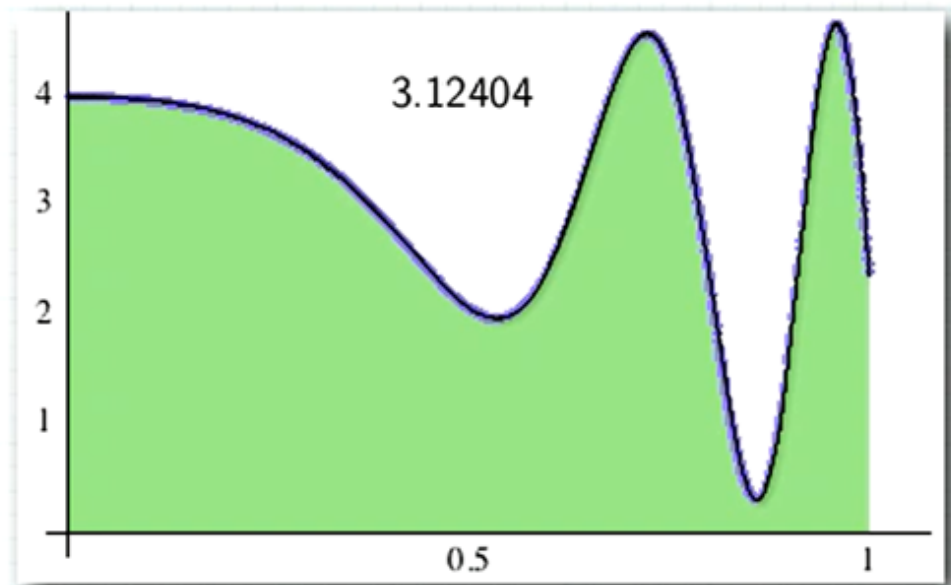
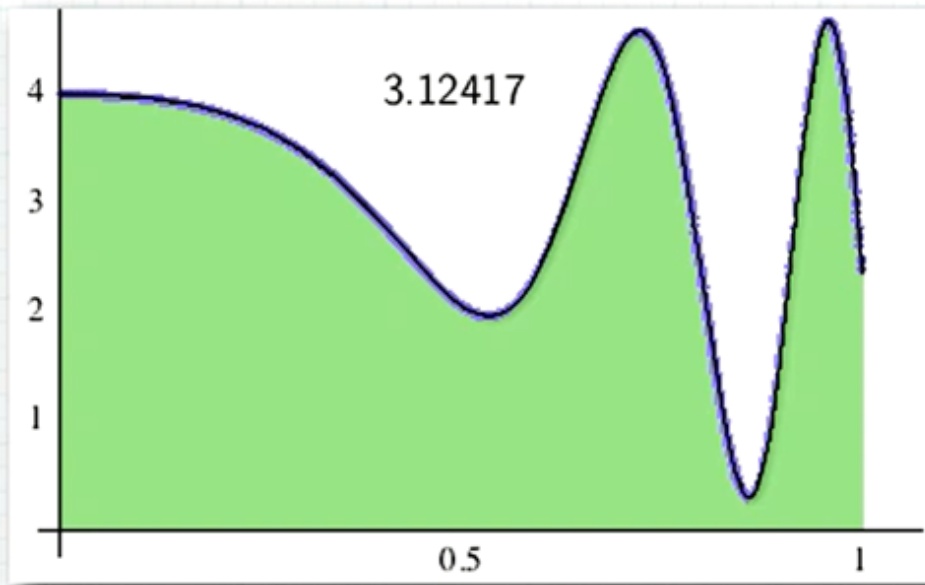


The **definite integral** really is calculating the **net area** under a curve over a given interval.

## **Theorem: The Existence of Definite Integrals**

**All continuous functions are integrable. That is, if a function is continuous on an interval  $[a,b]$ , then its definite integral over  $[a,b]$  exists.**

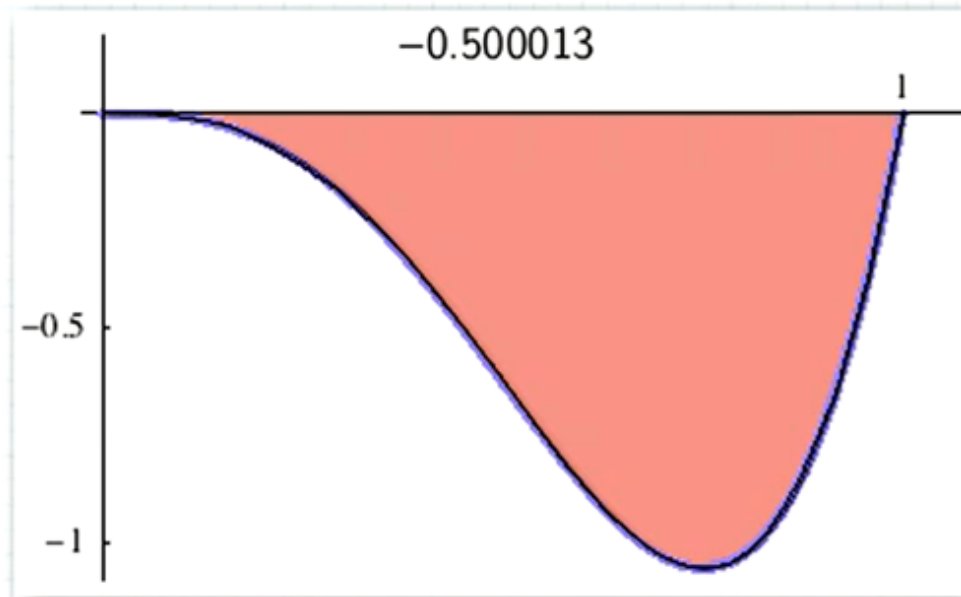
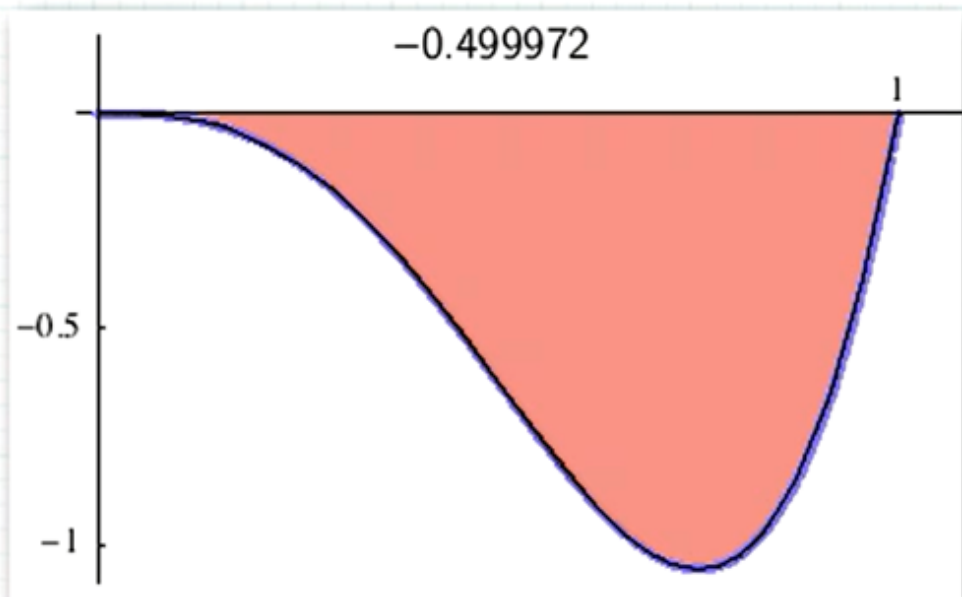
## Integrals and Signed Area



Let  $f$  be continuous on  $[a, b]$ . If  $f(x) \geq 0$  for all  $x$  in  $[a, b]$ , and if  $f(x) > 0$  for *some*  $x$  in  $[a, b]$ , then

$$\int_a^b f(x) dx > 0$$

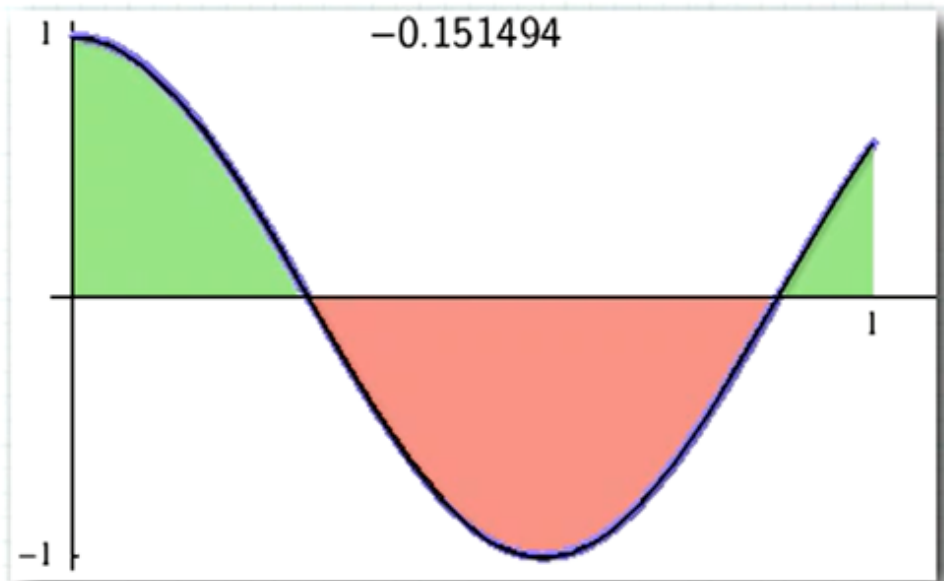
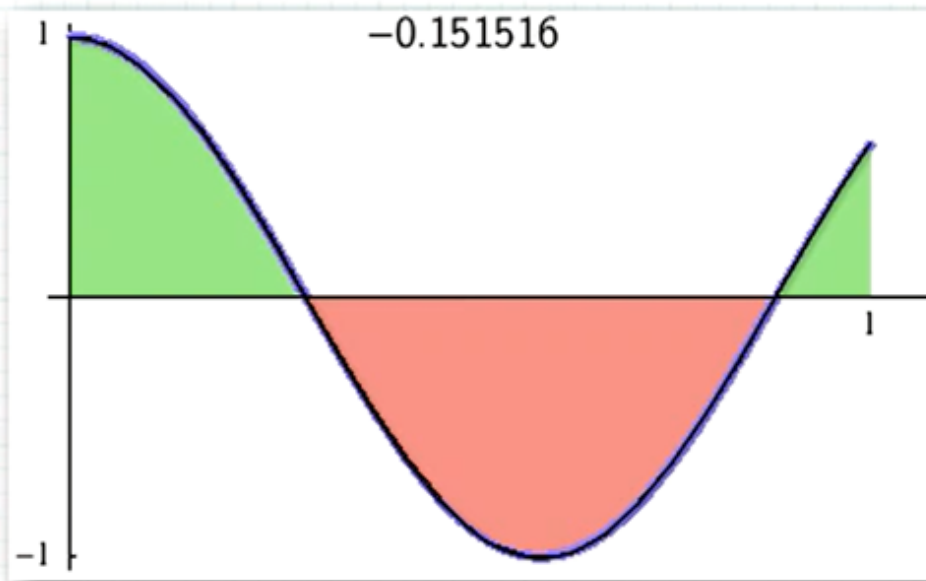
and equals the area of the region bounded by the graph of  $f$  and the  $x$ -axis between  $x = a$  and  $x = b$ .



Let  $f$  be continuous on  $[a, b]$ . If  $f(x) \leq 0$  for all  $x$  in  $[a, b]$ , and if  $f(x) < 0$  for *some*  $x$  in  $[a, b]$ , then

$$\int_a^b f(x) dx < 0$$

and  $-\int_a^b f(x) dx$  equals the area of the region bounded by the graph of  $f$  and the  $x$ -axis between  $x = a$  and  $x = b$ .



$\int_a^b f(x) dx$  equals the *difference* between the area under the graph of  $f$  above the  $x$ -axis and the area above the graph of  $f$  below the  $x$ -axis between  $x = a$  and  $x = b$ .

This is the **signed** (or *net*) **area** of the region bounded by the graph of  $f$  and the  $x$ -axis between  $x = a$  and  $x = b$ .

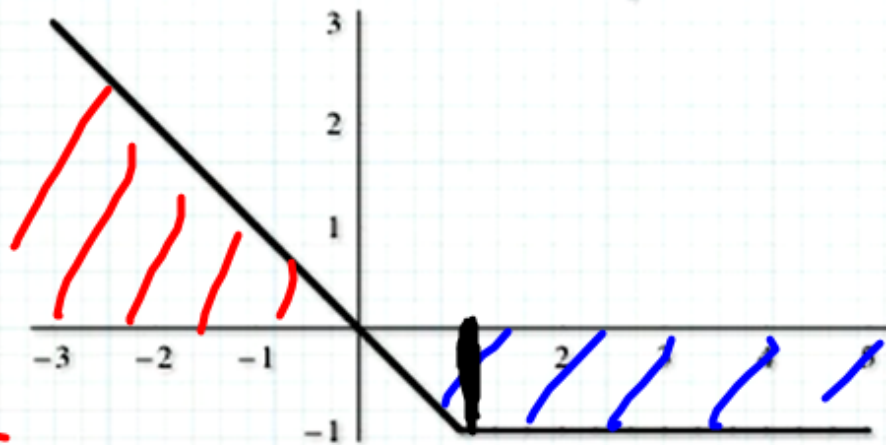
**Example** Let  $f(x) = \begin{cases} 1, & \text{if } x < 0 \\ 1 - x, & \text{if } 0 \leq x < 2. \\ x - 3, & \text{if } 2 \leq x \end{cases}$  Find  $\int_{-2}^5 f(x) dx$ .



$$\begin{aligned} \int_{-2}^5 f(x) dx &= (2)(1) + \frac{1}{2}(1)(1) - \frac{1}{2}(2)(1) \\ &\quad + \frac{1}{2}(2)(2) \\ &= 2 + \frac{1}{2} - 1 + 2 \\ &= \frac{7}{2} \end{aligned}$$



**Example** Let  $f(x) = \begin{cases} -x, & \text{if } x < 1 \\ -1, & \text{if } 1 \leq x \end{cases}$ . Find  $\int_{-3}^5 f(x) dx$ .



$$\int_{-3}^5 = \frac{1}{2}(3)(3) - \frac{1}{2}(1)(1) - (1)(4)$$
$$-3 = \frac{9}{2} - \frac{1}{2} - 4$$
$$= 0$$

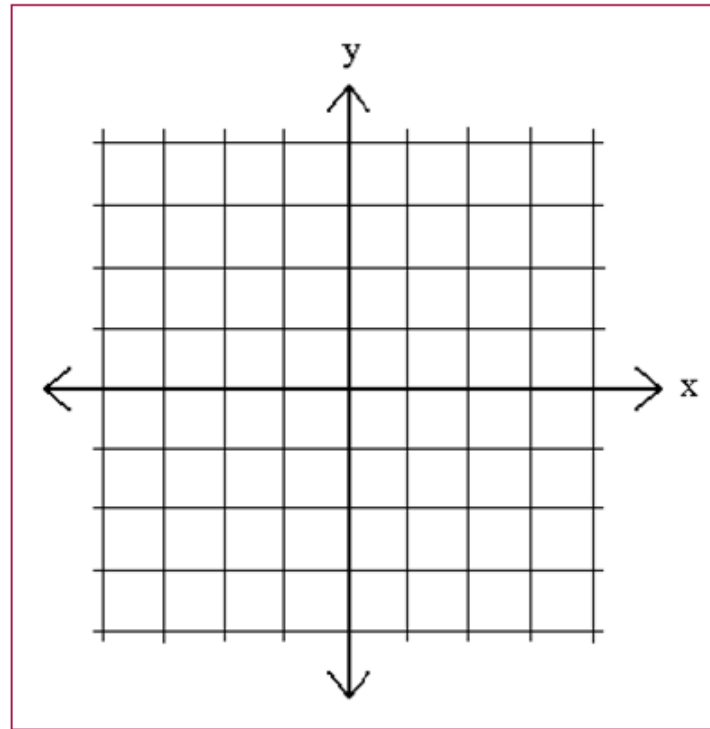
## Defn: Area Under a Curve

If  $y = f(x)$  is integrable over a closed interval  $[a, b]$  then the area under the curve  $y = f(x)$  from  $a$  to  $b$  is given by :

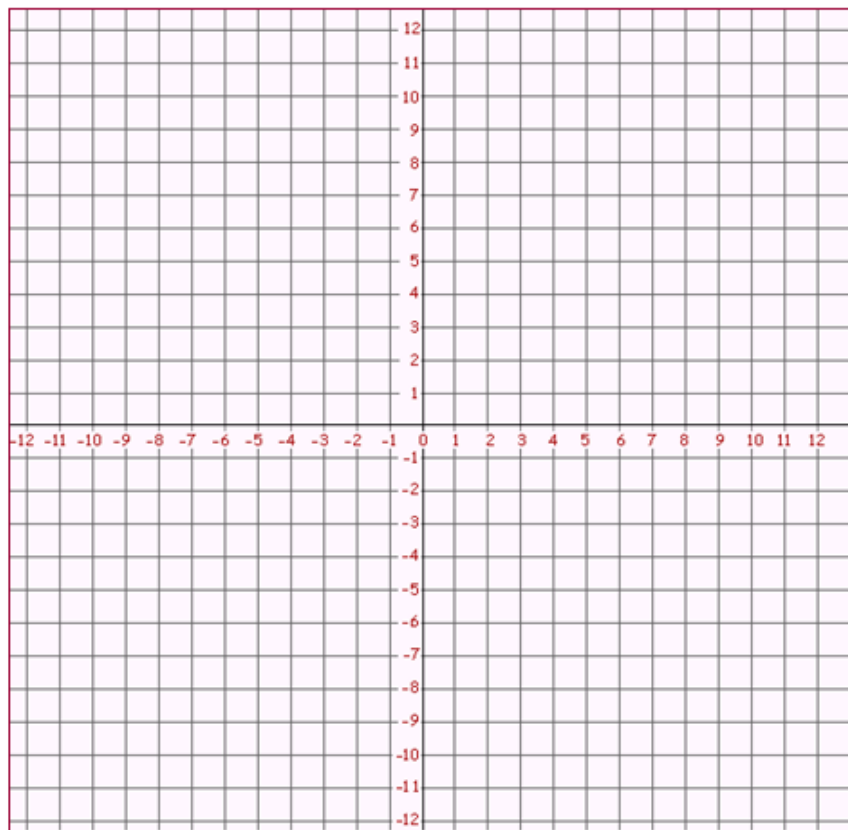
$$A = \int_a^b f(x) dx$$

**Ex.1 Evaluate the following integrals.**

a)  $\int_{-2}^2 \sqrt{4-x^2} dx$



$$\text{b) } \int_{-3}^6 2x dx$$



## Integrals of a Constant Function

If  $f(x) = c$ , where  $c$  is a constant on the interval

$$[a, b] \text{ then } \int_a^b f(x) dx = \int_a^b c dx = c(b - a)$$

## Integrals on a Calculator

We can use our graphing calculator to also evaluate definite integrals.

$$\int_a^b f(x)dx = \text{fnInt}(f(x), x, a, b)$$

**Ex.2 Evaluate using a calculator:**

$$\text{a) } \int_{-1}^2 x \sin x dx$$

$$\int_a^b f(x) dx = \text{fnInt}(f(x), x, a, b)$$

$$\text{b) } \int_0^5 e^{-x^2} dx$$

$$\text{c) } \int_0^1 \frac{4}{1+x^2} dx$$

# Assignment

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#'s 1-12,

13-19(area of a trapezoid),21-25

39,40 (use a calculator for these  
two)