Unit 5 Integration

5.1 Integration By Parts

Integration by parts is a technique used to integrate a function which is a product of two other functions. What integration of parts does is trade in one integral for a second (hopefully easier) integral.

"**Integration by parts**" is based on the product rule:

$$\frac{d}{dx}\big(u(x)v(x)\big) = u(x)\,v'(x) + v(x)\,u'(x)$$

Now rearrange:

$$u(x)\,v'(x) = \frac{d}{dx}\big(u(x)v(x)\big) - u'(x)\,v(x)$$

and antidifferentiate:

$$\int u(x)\,\underbrace{v'(x)dx}_{dv} = u(x)v(x) - \int v(x)\,\underbrace{u'(x)dx}_{du}$$

Here's the abbreviated form:

$$\boxed{\int u\,dv = u\,v - \int v\,du}$$

Example 1: $\int xsinxdx$

let $u = x$ $\quad v = -\cos x$

$du = dx \quad dv = \sin x\, dx$

$$\int x\sin x\,dx = uv - \int v\,du$$

$$= x(-\cos x) - \int -\cos x\,dx$$

$$\boxed{= -x\cos x + \sin x + C}$$

What to chose for “u”? LIPET (in general). Our first choice is the natural logarithm (L), if there is one. Next we look for an inverse trig function (I). Then we look for a polynomial (P). Then, look for an exponential (E) or a trigonometric function (T).

In general, we want “u” to be something that simplifies when differentiated, and dv to be something that remains manageable when integrated.

Example 2: $\int x\cos x\,dx$

let $u = x \qquad v = \sin x$

$du = dx \qquad dv = \cos x\,dx$

$$\int x\cos x\,dx = x\sin x - \int \sin x\,dx$$

$$= x\sin x + \cos x + c$$

Example 3: $\int(x^3 - 6x^2 + 2)lnxdx$

let $u = \ln x$ $\qquad v = \frac{x^4}{4} - 2x^3 + 2x$

$du = \frac{1}{x}dx \qquad dv = (x^3 - 6x^2 + 2)dx$

$$\int(x^3-6x^2+2)\ln x\,dx = \ln x\left(\frac{x^4}{4}-2x^3+2x\right) - \int\left(\frac{x^4}{4}-2x^3+2x\right)\cdot\frac{1}{x}dx$$

$$= \ln x\left(\frac{x^4}{4}-2x^3+2x\right) - \int\left(\frac{x^3}{4}-2x^2+2\right)dx$$

$$= \ln x\left(\frac{x^4}{4}-2x^3+2x\right) - \frac{x^4}{16} + \frac{2x^3}{3} - 2x + C$$

Sometimes one needs to do integration by parts twice to get the desired answer.

Example 4: $\int x^2 sinx dx$

let $u = x^2$ $\quad v = -\cos x$

$du = 2x\,dx$ $\quad dv = \sin x\,dx$

$$\int x^2 \sin x\,dx = -x^2\cos x - \int -\cos x\,(2x)\,dx$$

$$= -x^2\cos x + \int 2x\cos x\,dx$$

let

$u = 2x$ $\quad v = \sin x$

$du = 2\,dx$ $\quad dv = \cos x\,dx$

$$= -x^2\cos x + 2x\sin x - 2\int \sin x\,dx$$

$$= -x^2\cos x + 2x\sin x + 2\cos x + C$$

x^2		$\sin x \, dx$
	$+$	
$2x$		$-\cos x$
	$-$	
2		$-\sin x$
	$+$	
0		$\cos x$

$$= -x^2\cos x + 2x\sin x + 2\cos x + C$$

Example 5: $\int ln x dx$

let $u = \ln x$ $\quad v = x$

$du = \frac{1}{x} dx \quad dv = dx$

$$\int \ln x\, dx = x\ln x - \int x \cdot \frac{1}{x}\, dx$$

$$= x\ln x - \int 1\, dx$$

$$= x\ln x - x + c$$

* Sometimes one needs to do integration by parts two or more times until we get like terms with what we started with.

Example 6: $\int sin(lnx)\,dx$

let $u = \sin(\ln x)$ $\quad v = x$

$du = \cos(\ln x) \cdot \frac{1}{x}$ $\quad dv = dx$

$$\int \sin(\ln x)\,dx = x\sin(\ln x) - \int \cancel{x} \cdot \cos(\ln x) \cdot \cancel{\frac{1}{x}}\,dx$$

let

$u = \cos(\ln x)$ $\quad v = x$

$du = -\sin(\ln x) \cdot \frac{1}{x}\,dx$ $\quad dv = dx$

$$\int \sin(\ln x)\,dx = x\sin(\ln x) - \left[x\cos(\ln x) - \int \frac{-\cancel{x}\sin(\ln x)}{\cancel{x}}\,dx\right]$$

$$\int \sin(\ln x)\,dx = x\sin(\ln x) - x\cos(\ln x) - \underline{\int \sin(\ln x)\,dx}$$

$$2\int \sin(\ln x)\,dx = x\sin(\ln x) - x\cos(\ln x)$$

$$\int \sin(\ln x)\,dx = \frac{1}{2}\left(x\sin(\ln x) - x\cos(\ln x)\right)$$

Integration By Parts With Definite Integrals

Example 7: $\int_0^{\frac{\pi}{2}} 2x\sin x\,dx$

let $u = 2x$ $\qquad$ $v = -\cos x$

$du = 2\,dx$ $\qquad$ $dv = \sin x\,dx$

$$= -2x\cos x\Big|_0^{\pi/2} - \int_0^{\pi/2} -2\cos x\,dx$$

$$= -2x\cos x\Big|_0^{\pi/2} + 2\left[\sin x\Big|_0^{\pi/2}\right] = 2$$

$$= \cancel{-2\left(\tfrac{\pi}{2}\right)\cos\left(\tfrac{\pi}{2}\right)} - \cancel{\left(-2(0)\cos 0\right)} + 2\left[\sin\tfrac{\pi}{2} - \sin 0\right]$$

Assignment

Handout

#’s 1-6

5.1 Integration By Parts (a^x, e^x)

Example 1: $\int xe^x dx$

let $u = x$ $\quad v = e^x$

$du = dx$ $\quad dv = e^x dx$

$= xe^x - \int e^x dx$

$= xe^x - e^x + C$

Example 2: $\int xe^{-2x}\,dx$

let $u = x$ $\quad v = -\frac{1}{2}e^{-2x}$

$du = dx$ $\quad dv = e^{-2x}\,dx$

$= -\frac{1}{2}xe^{-2x} - \int -\frac{1}{2}e^{-2x}\,dx$

$= -\frac{1}{2}xe^{-2x} + \frac{1}{2}\int e^{-2x}\,dx$

$= -\frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x} + C$

$\frac{e^{-2x}}{-2}$

Example 3: $\int x^2 e^x dx$

Example 4: $\int x^3 e^x dx$

let $u = x^3$ $\quad v = e^x$

$du = 3x^2 dx \quad dv = e^x dx$

$= x^3 e^x - \int e^x (3x^2) dx$

$= x^3 e^x - 3 \int x^2 e^x dx$

$u = x^2 \quad v = e^x$

$du = 2x dx \quad dv = e^x dx$

$= x^3 e^x - 3 \left[x^2 e^x - \int 2x e^x dx \right]$

$$= x^3e^x - 3x^2e^x + 6\int xe^x\,dx$$

$u = x \qquad v = e^x$

$du = dx \qquad dv = e^x\,dx$

$$= x^3e^x - 3x^2e^x + 6\left[xe^x - \int e^x\,dx\right]$$

$$= \boxed{x^3e^x - 3x^2e^x + 6xe^x - 6e^x + C}$$

$$\int x^3 e^x \, dx$$

x^3		e^x
	$+$	
$3x^2$		e^x
	$-$	
$6x$		e^x
	$+$	
6		e^x
	$-$	
0		e^x

$$x^2e^x - 3x^2e^x + 6xe^x - 6e^x + C$$

Show table method with last example!

Example 5: $\int e^{x} cosx dx$

let $u = e^{x}$ $\quad v = \sin x$

$du = e^{x} dx$ $\quad dv = \cos x\, dx$

$= e^{x}\sin x - \int e^{x}\sin x\, dx$

let $u = e^{x}$ $\quad v = -\cos x$

$du = e^{x} dx$ $\quad dv = \sin x\, dx$

$$\int e^{x}\cos x\, dx = e^{x}\sin x - \left[-e^{x}\cos x - \int -e^{x}\cos x\, dx\right]$$

$$\underline{\int e^{x}\cos x\, dx} = e^{x}\sin x + e^{x}\cos x - \underline{\int e^{x}\cos x\, dx}$$

$$2\int e^x \cos x\,dx = e^x \sin x + e^x \cos x$$

$$\int e^x \cos x\,dx = \frac{1}{2}\left[e^x \sin x + e^x \cos x\right] + C$$

Example 6: $\int e^{-3x} sinx dx$

$u = e^{-3x}$ $v = -\cos x$

$du = -3e^{-3x}dx$ $dv = \sin x\,dx$

$= -e^{-3x}\cos x - \int -\cos x(-3e^{-3x})dx$

$= -e^{-3x}\cos x - 3\int e^{-3x}\cos x\,dx$

$u = e^{-3x}$ $v = \sin x$

$du = -3e^{-3x}dx$ $dv = \cos x\,dx$

$$\int e^{-3x}\sin x\,dx = -e^{-3x}\cos x - 3\left[e^{-3x}\sin x - 3\int -e^{-3x}\sin x\,dx\right]$$

$$\int e^{-3x}\sin x\,dx = -e^{-3x}\cos x - 3e^{-3x}\sin x - 9\int e^{-3x}\sin x\,dx$$

$$10\int e^{-3x}\sin x\,dx = -e^{-3x}\cos x - 3e^{-3x}\sin x$$

$$\int e^{-3x}\sin x\,dx = \frac{1}{10}\left[-e^{-3x}\cos x - 3e^{-3x}\sin x\right] + C$$

Example 7: $\int 2^x sinxdx$

let $u = 2^x$ $\quad$ $v = -\cos x$

$du = 2^x \ln 2\, dx$ $\quad$ $dv = \sin x\, dx$

$= -2^x \cos x - \int -2^x \ln 2 \cos x\, dx$

$= -2^x \cos x + \ln 2 \int 2^x \cos x\, dx$

let $u = 2^x$ $\quad$ $v = \sin x$

$du = 2^x \ln 2\, dx$ $\quad$ $dv = \cos x\, dx$

$$\int 2^x \sin x\,dx = -2^x \cos x + \ln 2\left[2^x \sin x - \int \ln 2\, 2^x \sin x\,dx\right]$$

$$\underline{\int 2^x \sin x\,dx} = -2^x \cos x + \ln 2(2^x \sin x) - \underline{(\ln 2)^2 \int 2^x \sin x\,dx}$$

$$\int 2^x \sin x\,dx + (\ln 2)^2 \int 2^x \sin x\,dx = -2^x \cos x + (\ln 2)\,2^x \sin x$$

$$\left(1 + (\ln 2)^2\right) \int 2^x \sin x\,dx = -2^x \cos x + (\ln 2)\,2^x \sin x$$

$$\int 2^x \sin x\,dx = \frac{1}{1 + (\ln 2)^2}\left[-2^x \cos x + (\ln 2)\,2^x \sin x\right] + C$$

Assignment

Handout

#'s 1-5

Handout Kuta Software Integration By Parts

#9 do long division in second integral