

4.10 Implicit Differentiation

P. 211 1-30

1. $3x - 4y - 12 = 0$

$$\frac{d}{dx}(3x - 4y - 12) = \frac{d}{dx}(0)$$

$$3 - 4 \frac{dy}{dx} - 0 = 0$$

$$3 - 4 \frac{dy}{dx} = 0$$

$$-4 \frac{dy}{dx} = -3$$

$$\frac{dy}{dx} = \left(\frac{3}{4}\right)$$

2. $x^2 + y = 4$

$$\frac{d}{dx}(x^2 + y) = \frac{d}{dx}(4)$$

$$2x + 1 \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \left(-2x\right)$$

3. $x^2 - y^2 = 4$

$$\frac{d}{dx}(x^2 - y^2) = \frac{d}{dx}(4)$$

$$2x - 2y \frac{dy}{dx} = 0$$

$$-2y \frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = \left(\frac{x}{y}\right)$$

4. $x^2 + y^2 - 25 = 0$

$$\frac{d}{dx}(x^2 + y^2 - 25) = \frac{d}{dx}(0)$$

$$2x + 2y \frac{dy}{dx} = 0 = 0$$

$$2y \frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = \left(-\frac{x}{y}\right)$$

5. $xy = 12$

$$\frac{d}{dx}(xy) = \frac{d}{dx}(12)$$

$$1(y) + x \frac{dy}{dx} = 0$$

$$x \frac{dy}{dx} = -y$$

$$\frac{dy}{dx} = \left(-\frac{y}{x}\right)$$

6. $\frac{x}{y} = 12$

$$x = 12y$$

$$\frac{d}{dx}(x) = \frac{d}{dx}(12y)$$

$$1 = 12 \frac{dy}{dx}$$

$$\left(\frac{1}{12}\right) = \frac{dy}{dx} \quad \text{or} \quad \left(\frac{y}{x}\right)$$

(2 answers)

$$\begin{aligned} y - x^2 &= 0 \\ y^2 &= x^2 \\ y - x^2 &= 0 \\ \frac{dy}{dx} - 2x &= 0 \\ \frac{dy}{dx} &= \frac{2x}{1} \\ &= \frac{y}{x} \end{aligned}$$

4.10 - Continued

7. $\frac{1}{x} + \frac{1}{y} = 10$

$$\frac{d}{dx} \left(x^{-1} + y^{-1} \right) = \frac{d}{dx} (10)$$

$$-1x^{-2} + -1(y^{-2}) \frac{dy}{dx} = 0$$

$$\frac{-1}{x^2} - \frac{1}{y^2} \frac{dy}{dx} = 0$$

$$-\frac{1}{y^2} \frac{dy}{dx} = \frac{1}{x^2}$$

$$\frac{dy}{dx} = \frac{-y^2}{x^2}$$

8. $x^2 + y^2 - 8x + 6y - 30 = 0$

$$\frac{d}{dx} (x^2 + y^2 - 8x + 6y - 30) = \frac{d}{dx} (0)$$

$$2x + 2y \frac{dy}{dx} - 8 + 6 \frac{dy}{dx} - 0 = 0$$

$$2x - 8 + (2y + 6) \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-2x + 8}{2y + 6}$$

$$\frac{dy}{dx} = \frac{-x + 4}{y + 3}$$

9. $x^2 + 3xy = 30$

$$\frac{d}{dx} (x^2 + 3xy) = \frac{d}{dx} (30)$$

$$2x + 3y + 3x \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-2x - 3y}{3x}$$

10. $x - 2xy = 2y$

$$\frac{d}{dx} (x - 2xy) = \frac{d}{dx} (2y)$$

$$1 - (2y + 2x \frac{dy}{dx}) = 2 \frac{dy}{dx}$$

$$1 - 2y - 2x \frac{dy}{dx} = 2 \frac{dy}{dx}$$

$$1 - 2y = (2x + 2) \frac{dy}{dx}$$

$$\frac{1 - 2y}{2x + 2} = \frac{dy}{dx}$$

11. $x^2 - y^2 - 2xy - 8 = 0$

$$\frac{d}{dx} (x^2 - y^2 - 2xy - 8) = \frac{d}{dx} (0)$$

$$2x - 2y \frac{dy}{dx} - (2y + 2x \frac{dy}{dx}) - 0 = 0$$

$$2x - 2y \frac{dy}{dx} - 2y - 2x \frac{dy}{dx} = 0$$

$$2x - 2y = (2x + 2y) \frac{dy}{dx}$$

$$\frac{2x - 2y}{2x + 2y} = \frac{dy}{dx}$$

$$\frac{2(x - y)}{2(x + y)} = \frac{dy}{dx}$$

$$\frac{x - y}{x + y} = \frac{dy}{dx}$$

4.10 Continued

12. $(x-2)^2 - (y+3)^2 = 1$

$x^2 - 4x + 4 - y^2 - 6y - 9 = 0$

$x^2 - y^2 - 4x - 6y - 5 = 0$

$\frac{d}{dx}(x^2 - y^2 - 4x - 6y - 5) = 0$

$2x - 2y \frac{dy}{dx} - 4 - 6 \frac{dy}{dx} - 0 = 0$

$\frac{dy}{dx}(-2y - 6) = -2x + 4$

$\frac{dy}{dx} = \frac{-2x + 4}{-2y - 6} = \frac{-1(x-2)}{-1(y+3)} = \frac{x-2}{y+3}$

13. $x = \frac{1}{4}(y-5)^2 + 3$

$x = \frac{1}{4}(y^2 - 10y + 25) + 3$

$4x = y^2 - 10y + 37$

$\frac{d}{dx}(4x) = \frac{d}{dx}(y^2 - 10y + 37)$

$4 = 2y \frac{dy}{dx} - 10 \frac{dy}{dx}$

$\frac{4}{2y - 10} = \frac{dy}{dx}$

$\frac{2}{y-5} = \frac{dy}{dx}$

14. $2x^2 - 5xy + y^2 = 20$

$\frac{d}{dx}(2x^2 - 5xy + y^2) = \frac{d}{dx}(20)$

$4x - (5y + 5x \frac{dy}{dx}) + 2y \frac{dy}{dx} = 0$

$4x - 5y - 5x \frac{dy}{dx} + 2y \frac{dy}{dx} = 0$

$\frac{dy}{dx}(-5x + 2y) = -4x + 5y$

$\frac{dy}{dx} = \frac{-4x + 5y}{-5x + 2y}$

or $\frac{dy}{dx} = \frac{4x - 5y}{5x - 2y}$

$\frac{dy}{dx} = \frac{4x - 5y}{5x - 2y}$

15. $\sqrt{x} - \sqrt{y} = 4$

$\frac{d}{dx}(x^{1/2} - y^{1/2}) = \frac{d}{dx}(4)$

$\frac{1}{2}x^{-1/2} - \frac{1}{2}y^{-1/2} \frac{dy}{dx} = 0$

$\frac{1}{2\sqrt{x}} = \frac{1}{2\sqrt{y}} \frac{dy}{dx} = \frac{1}{2\sqrt{x}}$

$\frac{dy}{dx} = \frac{2\sqrt{y}}{2\sqrt{x}} = \frac{\sqrt{y}}{\sqrt{x}}$

4.10- Continued

16. $(x^2 - y^2)^2 = 6xy$

$$x^4 - 2x^2y^2 + y^4 = 6xy$$

$$\frac{d}{dx} (x^4 - 2x^2y^2 + y^4) = \frac{d}{dx} (6xy)$$

$$4x^3 - (4xy^2 + 2x^2 \frac{dy}{dx}) + 4y^3 \frac{dy}{dx} = 6y + 6x \frac{dy}{dx}$$

$$4x^3 - 4xy^2 - 4x^2y \frac{dy}{dx} + 4y^3 \frac{dy}{dx} = 6y + 6x \frac{dy}{dx}$$

$$\frac{dy}{dx} (-4x^2y + 4y^3 - 6x) = (6y - 4x^3 + 4xy^2)$$

$$\frac{dy}{dx} = \frac{3y - 2x^3 + 2xy^2}{-2xy^2 + 2y^3 - 3x} \quad \text{or} \quad \frac{2x^3 - 2xy^2 - 3y}{-2y^3 + 2x^2y + 3x}$$

17. $\frac{x^2}{x-y^2} = 11$

$$\frac{d}{dx} \left(\frac{x^2}{x-y^2} \right) = \frac{d}{dx} (11)$$

$$\frac{2x(x-y^2) - (x^2)(1-2y \frac{dy}{dx})}{(x-y^2)^2} = 0$$

$$2x^2 - 2xy^2 - x^2 + 2x^2y \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-x^2 + 2xy^2}{2x^2y}$$

$$\frac{dy}{dx} = \frac{-x + 2y^2}{2xy}$$

or $x^2 = 11x - 11y^2$

$$\frac{d}{dx} (x^2) = \frac{d}{dx} (11x - 11y^2)$$

$$2x = 11 - 22y \frac{dy}{dx}$$

$$\frac{2x - 11}{-22y} = \frac{dy}{dx}$$

$$\frac{-2x + 11}{22y} = \frac{dy}{dx}$$

18. $\frac{x}{y} + \frac{y}{x} = 5$

$$x^2 + y^2 = 5xy$$

$$\frac{d}{dx} (x^2 + y^2) = \frac{d}{dx} (5xy)$$

$$2x + 2y \frac{dy}{dx} = 5y + 5x \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{5y - 2x}{2y - 5x} \quad \text{or} \quad \frac{2x - 5y}{5x - 2y}$$

4.10 Continued

19. $\frac{x+y}{x-y} = \frac{1}{y}$

$$x-y = xy + y^2$$

$$\frac{d}{dx}(x-y) = \frac{d}{dx}(xy + y^2)$$

$$1 - \frac{dy}{dx} = y + x\frac{dy}{dx} + 2y\frac{dy}{dx}$$

$$1 - y = (1 + x + 2y)\frac{dy}{dx}$$

$$\frac{1-y}{1+x+2y} = \frac{dy}{dx}$$

20. $x^3 - 2y^2x - 3y = 7$

$$\frac{d}{dx}(x^3 - 2y^2x - 3y) = \frac{d}{dx}(7)$$

$$3x^2 - (4yx\frac{dy}{dx} + 2y^2) - 3\frac{dy}{dx} = 0$$

$$3x^2 - 4xy\frac{dy}{dx} - 2y^2 - 3\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{2y^2 - 3x^2}{-4xy - 3} = \frac{3x^2 - 2y^2}{4xy + 3}$$

21. $\sqrt{x^2 + y^2} = x^2$

$$(x^2 + y^2)^{\frac{1}{2}} = x^2$$

$$x^2 + y^2 = x^4$$

$$2x + 2y\frac{dy}{dx} = 4x^3$$

$$2y\frac{dy}{dx} = 4x^3 - 2x$$

$$\frac{dy}{dx} = \frac{2x^3 - x}{y}$$

4.10 - Continued

22. $x^3 + 2y^2 = 10y$ (2,1)

$$\frac{d}{dx}(x^3 + 2y^2) = \frac{d}{dx}(10y)$$

$$3x^2 + 4y \frac{dy}{dx} = 10 \frac{dy}{dx}$$

$$3(2)^2 + 4(1) \frac{dy}{dx} = 10 \frac{dy}{dx}$$

$$\frac{12}{6} = \frac{dy}{dx}$$

$$\boxed{2} = \frac{dy}{dx}$$

23. $x^2 - 3xy - y = 5$ (-1,2)

$$\frac{d}{dx}(x^2 - 3xy - y) = \frac{d}{dx}(5)$$

$$2x - (3y + 3x \frac{dy}{dx}) - \frac{dy}{dx} = 0$$

$$2x - 3y - 3x \frac{dy}{dx} - \frac{dy}{dx} = 0$$

$$2x - 3y = (3x+1) \frac{dy}{dx}$$

$$\frac{2x-3y}{3x+1} = \frac{dy}{dx}$$

$$\frac{2(-1) - 3(2)}{3(-1) + 1} = \frac{dy}{dx}$$

$$\frac{-8}{-2} = \boxed{4} = \frac{dy}{dx}$$

24. $x^2y + xy^2 + 6 = 0$ (3,-1)

$$\frac{d}{dx}(x^2y + xy^2 + 6) = \frac{d}{dx}(0)$$

$$2xy + x^2 \frac{dy}{dx} + y^2 + x(2y) \frac{dy}{dx} + 0 = 0$$

$$2xy + y^2 = (-x^2 - 2xy) \frac{dy}{dx}$$

$$\frac{(2xy + y^2)}{(-x^2 - 2xy)} = \frac{dy}{dx}$$

$$\frac{2(3)(-1) + (-1)^2}{-(3)^2 - 2(3)(-1)} = \frac{-6+1}{-9+6} = \frac{-5}{-3}$$

$$\frac{dy}{dx} = \boxed{\frac{5}{3}}$$

$$\text{or } \frac{dy}{dx} = \frac{-2xy - y^2}{(x^2 + 2xy)}$$

$$= \frac{-2(3)(-1) - (-1)^2}{(3)^2 + 2(3)(-1)}$$

$$= \frac{6-1}{9-6} = \boxed{\frac{5}{3}}$$

25. $(x+y)^2 - xy = 3$ (1,1)

$$x^2 + 2xy + y^2 - xy = 3$$

$$x^2 + xy + y^2 = 3$$

$$\frac{d}{dx}(x^2 + xy + y^2) = \frac{d}{dx}(3)$$

$$2x + y + x \frac{dy}{dx} + 2y \frac{dy}{dx} = 0$$

$$2x + y = (-x - 2y) \frac{dy}{dx}$$

$$\frac{2x+y}{-x-2y} = \frac{dy}{dx}$$

$$\frac{2(1)+1}{-1-2(1)} = \boxed{-1}$$

4.10 - Continued

26. $y^2 - \frac{x}{2y} = 8 \quad (6, 3)$

$2y^3 - x = 16y$

$\frac{d}{dx}(2y^3 - x) = \frac{d}{dx}(16y)$

$6y^2 \frac{dy}{dx} - 1 = 16 \frac{dy}{dx}$

$(6y^2 - 16) \frac{dy}{dx} = 1$

$\frac{dy}{dx} = \frac{1}{6y^2 - 16} = \frac{1}{6(3)^2 - 16}$

$\frac{dy}{dx} = \left(\frac{1}{38}\right)$

27. $\frac{x^2 + y^2}{x - y} = -10 \quad (-4, -2)$

$x^2 + y^2 = -10x + 10y$

$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(-10x + 10y)$

$2x + 2y \frac{dy}{dx} = -10 + 10 \frac{dy}{dx}$

$2x + 10 = (-2y + 10) \frac{dy}{dx}$

$\frac{2x + 10}{-2y + 10} = \frac{dy}{dx}$

$\frac{x + 5}{-y + 5} = \frac{dy}{dx} \quad \frac{-4 + 5}{-(-2) + 5} = \left(\frac{1}{7}\right)$

28. $x^2 - y^2 = 1 \quad \left(\frac{5}{4}, \frac{3}{4}\right)$

(1) Explicit Differentiation
(solve for y)

$y = \sqrt{x^2 - 1}$
 $y' = \frac{1}{2}(x^2 - 1)^{-1/2} \cdot 2x$

$y' = x(x^2 - 1)^{-1/2}$
 $y' = \frac{5}{4} \left(\left(\frac{5}{4}\right)^2 - 1\right)^{-1/2}$

$y' = \frac{5}{4} \left(\frac{9}{16}\right)^{-1/2}$
 $y' = \frac{5}{4} \left(\frac{4}{3}\right) = \left(\frac{5}{3}\right)$

(2) Implicit Differentiation

$\frac{d}{dx}(x^2 - y^2) = \frac{d}{dx}(1)$

$2x - 2y \frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{x}{y}$

$\frac{dy}{dx} = \frac{\frac{5}{4}}{\frac{3}{4}} = \left(\frac{5}{3}\right)$

29. $\frac{x^2}{25} + \frac{y^2}{9} = 1 \quad \left(4, \frac{4}{5}\right)$

$\frac{d}{dx}\left(\frac{x^2}{25} + \frac{y^2}{9}\right) = \frac{d}{dx}(1)$

$\frac{2x}{25} + \frac{2y}{9} \frac{dy}{dx} = 0$

$\frac{2x}{25} = -\frac{2y}{9} \frac{dy}{dx}$

$\frac{2x}{25} \left(\frac{9}{-2y}\right) = \frac{dy}{dx}$

$\frac{9x}{-25y} = \frac{dy}{dx}$

$\frac{9(4)}{-25(\frac{4}{5})} = \frac{dy}{dx}$

$\left(\frac{-4}{5}\right) = \frac{dy}{dx}$

$y - y_1 = m(x - x_1)$

$y - \frac{4}{5} = -\frac{4}{5}(x - 4)$

$5\left(y - \frac{4}{5}\right) = -4(x - 4)$

$5y - 4 = -4x + 16$

$4x + 5y - 20 = 0$

4.10 Continued

30. $x^2 - 2xy - y^2 + 17 = 0$ (2,3)

$$\frac{d}{dx}(x^2 - 2xy - y^2 + 17) = \frac{d}{dx}(0)$$

$$2x - (2y + 2x \frac{dy}{dx}) - 2y \frac{dy}{dx} = 0$$

$$2x - 2y - 2x \frac{dy}{dx} - 2y \frac{dy}{dx} = 0$$

$$(2x - 2y) = \frac{dy}{dx} (2x + 2y)$$

$$\frac{2x - 2y}{2x + 2y} = \frac{dy}{dx}$$

$$\frac{x - y}{x + y} = \frac{dy}{dx}$$

$$\frac{2 - 3}{2 + 3} = \frac{dy}{dx}$$

$$\frac{-1}{5} = \frac{dy}{dx}$$

$$y - y_1 = m(x - x_1)$$

$$y - 3 = -\frac{1}{5}(x - 2)$$

$$5(y - 3) = -1(x - 2)$$

$$5y - 15 = -1x + 2$$

$$\boxed{x + 5y - 17 = 0}$$