4.6 Related Rates

4.6 Related Rates

Learning Target:

SWBAT solve a number of related rate problems using impicit differentiation.

We have seen how the Chain Rule can be used to find a derivative implicitly. Another important application of the Chain Rule is to find rates of change of two or more related variables that are changing with respect to time.

Ex.1 A pebble is dropped into a calm pond causing ripples to form in the shape of concentric circles. The radius of the outside ripple is increasing at a rate of 1.5 m/s. When the radius is exactly 4 metres, at what rate is the area of the ripple changing?

$$\frac{dr}{dt} = 1.5\,m/s$$

$$r = 4\,m$$

$$\frac{dA}{dt} = ?$$

$$A = \pi r^2$$

$$\boxed{\frac{dA}{dt}} = 2\pi \boxed{r} \boxed{\frac{dr}{dt}}$$

?

$$\frac{dA}{dt} = 2\pi(4m)(1.5m/s)$$

$$\frac{dA}{dt} = 12\pi \ m^2/s$$

http://www.youtube.com/watch?v=WIBpPpE-XRg&feature=related

Ex.2 A spherical snowball is melting in such a way that its volume is decreasing at a rate of 1cm³/min. At what rate is the radius decreasing when the radius is 5 cm?

$$\frac{dV}{dt} = -1\,cm^3/min$$

$$\frac{dr}{dt} = ?$$

$$r = 5cm$$

$$V = \frac{4}{3}\pi r^3$$

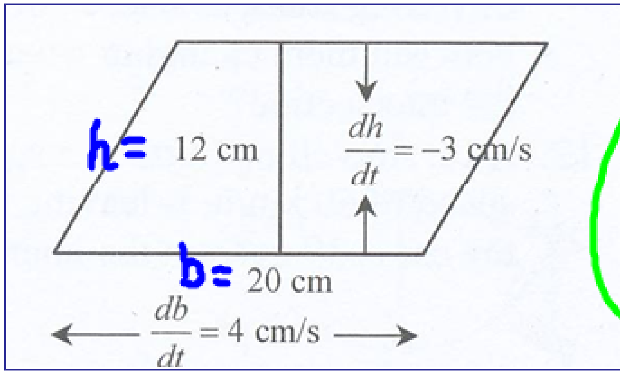
$$\frac{dV}{dt} = 4\pi r^2 \boxed{\frac{dr}{dt}}$$

$$\frac{\frac{dV}{dt}}{4\pi r^2} = \frac{dr}{dt}$$

$$\frac{-1\ cm^3/min}{4(\pi)(5cm)^2} = \frac{dr}{dt}$$

$$\frac{-1\ cm^{\cancel{3}}/min}{100\pi\ \cancel{cm^2}} = dr/dt = \frac{-1}{100\pi}\ cm/min$$

Example 3 The base of a parallelogram is growing at a rate of 4 cm/s while its height is shrinking at a rate of 3 cm/s. How is the area of the parallelogram changing when the base is 20 cm and the height is 12 cm?



$$\frac{dA}{dt} = \underline{b}\,\underline{\frac{dh}{dt}} + \underline{h}\,\underline{\frac{db}{dt}}$$

$$\boxed{A = bh}$$

$$\frac{dA}{dt} = (20\,\text{cm})\left(-3\,\frac{\text{cm}}{\text{s}}\right) + (12\,\text{cm})(4\,\text{cm/s})$$

$$\frac{dA}{dt} = -60\,\frac{\text{cm}^2}{\text{s}} + 48\,\text{cm}^2/\text{s}$$

$$dA/dt = -12\,\text{cm}^2/\text{s}$$

Ex.4 A spherical balloon increases in diameter at a rate of 10 cm/min. Find the rate of increase of the surface area of the sphere at the instant the surface area is 4π m^2.

$\boxed{S = 4\pi r^2}$

$\boxed{\frac{dS}{dt}} = 8\pi r \frac{dr}{dt}$ ✓

~~$\frac{dd}{dt} = 10 \text{ cm/min}$~~

$\frac{dr}{dt} = 5 \text{ cm/min}$

$\frac{dS}{dt} = ?$

$S = 4\pi m^2$

$S = 4\pi r^2$

$\frac{4\pi m^2}{\cancel{4\pi}} = \frac{\cancel{4\pi} r^2}{\cancel{4\pi}}$

$\sqrt{1 m^2} = \sqrt{r^2}$

$1 m = r = 100 \text{ cm}$

$$\frac{dS}{dt} = 8\pi r \frac{dr}{dt}$$

$$\frac{dS}{dt} = 8\pi(100\text{ cm})(5\text{ cm/min})$$

$$\frac{dS}{dt} = 4000\pi\text{ cm}^2/\text{min}$$

Formuals Needed

Area of a Circle: $A = \pi r^2$

Volume of a Sphere: $V = \frac{4}{3}\pi r^3$

Surface Area of a Sphere: $S = 4\pi r^2$

Volume of a cube: $V = x^3$

Circumference of a Circle: $C = 2\pi r$

Volume Cone: $V = \frac{1}{3}\pi r^2 h$

Asssignment
Calculus 30 Text Page 290
#'s 1, 3, 4, 7, 8, 10,

Cone Problems

A water tank is in the shape of an inverted cone. The height of the cone is 10 meters and the diameter of the base is 8 meters as shown in Figure 4.1-1. Water is being pumped into the tank at the rate of 2 m^3/min. How fast is the water level rising when the water is 5 meters deep? (See Figure 4.1-1 on page 130.)

$h_c = 10m$

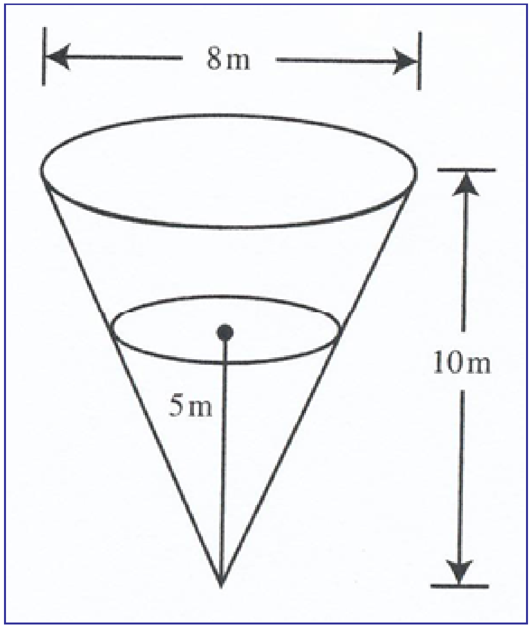
$r_c = 4m$

$V = \frac{1}{3}\pi r^2 h$

$\frac{dV}{dt} = 2m^3/min$

$\frac{dh}{dt} = ?$

$h_w = 5m$



Ratio

$$\frac{r_c}{h_c} = \frac{4}{10}$$

$$\frac{r_c}{h_c} = \frac{2}{5}$$

$$r = \frac{2h}{5}$$

$$V = \frac{1}{3}\pi r^2 h$$

$$V = \frac{1}{3}\pi\left(\frac{2h}{5}\right)^2 h$$

$$V = \frac{1}{3}\pi\left(\frac{4h^2}{25}\right)h$$

$$V = \frac{4\pi h^3}{75}$$

$$\frac{dV}{dt} = \frac{12\pi}{75}h^2\frac{dh}{dt}$$

$$2\,\frac{m^3}{min} = \frac{12\pi}{75}(5m)^2\,\frac{dh}{dt}$$

$$2\,\frac{m^{\cancel{3}}}{min} = \frac{12\pi}{\cancel{75}\,3}\left(\cancel{25}\,\cancel{m^2}\right)\frac{dh}{dt}$$

$$2\,\frac{m}{min} = (4\pi)\,\frac{dh}{dt}$$

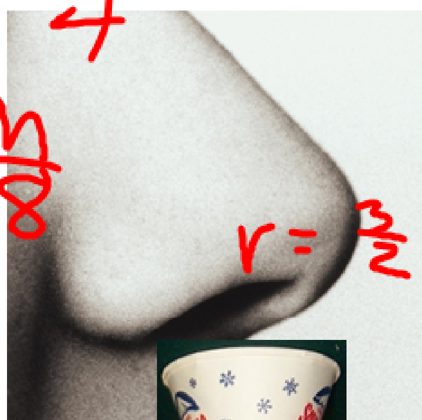
$$\frac{2}{4\pi}\,\frac{m}{min} = \frac{dh}{dt}$$

$$.159\ m/min = dh/dt$$

Example 3: The nightmare has come to pass. All of Kelley's extensive surgeries and nasal passage scrapings have (unfortunately) gone awry, and he waits in the ear, nose, and throat doctor's office waiting area spewing bloody nose drippings into a conical paper cup at the rate of 2.5 in^3/min. The cup is being held with the vertex down and has a height of 4 inches and a base of 3 inches. How fast is the mucous level rising in the cup when the "liquid" is 2 inches deep?

$\frac{r}{h} = \frac{3/2}{4}$

$r = \frac{3h}{8}$



$r = \frac{3}{2}$ in

$h_c = 4$ in

$V = \frac{1}{3}\pi\left(\frac{3h}{8}\right)^2 h$

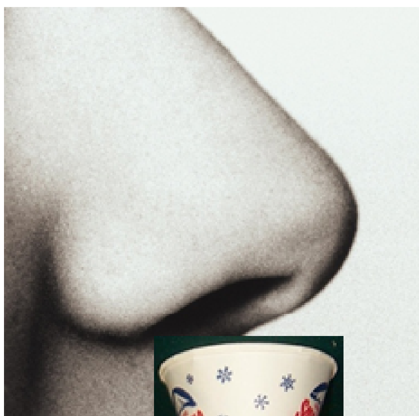
$V = \frac{9\pi h^3}{192}$

$\frac{dV}{dt} = \frac{27\pi h^2}{192}\frac{dh}{dt}$

$2.5 = \frac{27\pi(2)^2}{192}\, dh/dt$

1.415 in/min $= dh/dt$

Example 3: The nightmare has come to pass. All of Kelley's extensive surgeries and nasal passage scrapings have (unfortunately) gone awry, and he waits in the ear, nose, and throat doctor's office waiting area spewing bloody nose drippings into a conical paper cup at the rate of 2.5 in^3/min. The cup is being held with the vertex down and has a height of 4 inches and a base of 3 inches. How fast is the mucous level rising in the cup when the "liquid" is 2 inches deep?



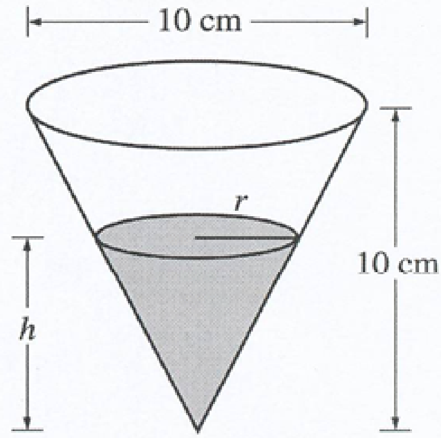
$$\frac{dh}{dt} = -\frac{3}{10}\ \text{cm/h}$$

$$\frac{r}{h} = \frac{5}{10}$$

$$r = \frac{h}{2}$$

AP® CALCULUS AB 2002 SCORING GUIDELINES

Question 5



A container has the shape of an open right circular cone, as shown in the figure above. The height of the container is 10 cm and the diameter of the opening is 10 cm. Water in the container is evaporating so that its depth h is changing at the constant rate of $\frac{-3}{10}$ cm/hr.

(The volume of a cone of height h and radius r is given by $V = \frac{1}{3}\pi r^2 h$.)

(a) Find the volume V of water in the container when $h = 5$ cm. Indicate units of measure.

(b) Find the rate of change of the volume of water in the container, with respect to time, when $h = 5$ cm. Indicate units of measure.

dV/dt

(c) Show that the rate of change of the volume of water in the container due to evaporation is directly proportional to the exposed surface area of the water. What is the constant of proportionality?

a) $V = \frac{1}{3}\pi r^2 h$

1

$$V = \frac{1}{3}\pi\left(\frac{5}{2}\text{cm}\right)^2(5\text{cm}) = \frac{1}{3}\pi\left(\frac{25}{4}\text{cm}^2\right)(5\text{cm}) = \frac{125\pi}{12}\text{cm}^3$$

b)

$\frac{dV}{dt} = ?$

$h = 5cm$

$\frac{dh}{dt} = -\frac{3}{10} cm/hr$

$V = \frac{1}{3}\pi r^2 h$

$V = \frac{1}{3}\pi \left(\frac{h}{2}\right)^2 h$

$V = \frac{1}{3}\pi \left(\frac{h^2}{4}\right) h$

$V = \frac{\pi h^3}{12}$

$\frac{dV}{dt} = \frac{\pi}{4} h^2 \frac{dh}{dt}$

$$\frac{dV}{dt} = \frac{\pi}{4}(5\,cm)^2\left(-\frac{3}{10}\frac{cm}{h}\right)$$

$$= \frac{\pi}{4}\left(25\,cm^2\right)\left(-\frac{3}{10}\frac{cm}{h}\right)$$

$$\frac{dV}{dt} = -\frac{75\pi}{40}\,cm^3/hr$$

c) $\frac{dV}{dt} \propto \pi r^2$

$\frac{dV}{dt} = k\pi r^2$ $k = -\frac{3}{10}$

$\frac{dV}{dt} = k\pi\left(\frac{h}{2}\right)^2$

$\frac{dV}{dt} = k\pi\left(\frac{h^2}{4}\right)$

$\frac{dh}{dt} = k$

$dV/dt = -300 \ in^3/min$

3. Sand is falling from a rectangular box container whose base measures 40 inches by 20 inches at a constant rate of 300 cubic inches per minute. (Include units in all your answers.)

$dh/dt = ?$

(a) How is the depth of the sand in the box changing?

(b) The sand is forming a conical pile $\left(V = \frac{\pi}{3}r^2h\right)$. At a particular moment, the pile is 23 inches high and the diameter of the base is 16 inches. The diameter of the base at this moment is increasing at 1.5 inches per minute. At this moment,

(i) how fast is the area of the circular base of the cone increasing?

(ii) how fast is the height of the pile increasing?

a) $V = (40)(20)h$

$V = 800h$

$dV/dt = 800 \ dh/dt$

$-300 = 800 \ dh/dt$

$-\frac{3}{8} \ in/min = dh/dt$

b) i) $\frac{dV}{dt} = 300 \text{ in}^3/\text{min}$

$\frac{dd}{dt} = 1.5 \text{ in/min}$

$\frac{dr}{dt} = 0.75 \text{ in/min}$

$\frac{dA}{dt} = ?$

$r = 8 \text{ in}$

$A = \pi r^2$

$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$

$= 2\pi (8 \text{ in})(.75 \frac{\text{in}}{\text{min}})$

$\frac{dA}{dt} = 12\pi \text{ in}^2/\text{min}$

ii) $\frac{dV}{dt} = 300 \text{ in}^3/\text{min}$

r = 8 in

h = 23 in

$\frac{r}{h} = \frac{8}{23}$

$r = \frac{8h}{23}$

$V = \frac{1}{3}\pi\left(\frac{8h}{23}\right)^2(h)$

$V = \frac{1}{3}\pi\left(\frac{64h^2}{529}\right)h$

$V = \frac{64\pi h^3}{1587}$

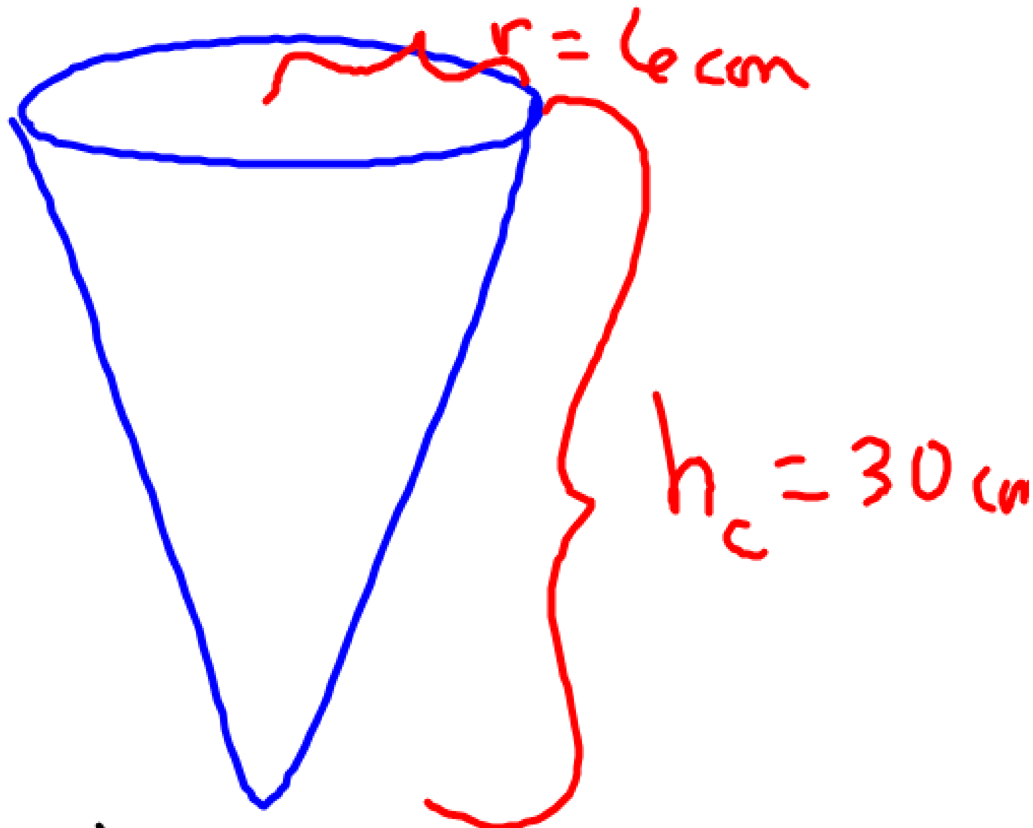
http://www.midnighttutor.com/rate_deriv_sand.html

A conical vase is 30 cm high with a radius of 6 cm. It is being filled at a rate of $10\ cm^3/s$, find the rate at which the water level is rising when the depth is 20 cm.

$\frac{dV}{dt} = 10\,cm^3/s$

$\frac{dh}{dt} = ?$

$h_w = 20cm$



$V = \frac{1}{3}\pi r^2 h$

We want Volume in terms of just h.

Ratio

$\frac{r_c}{h_c} = \frac{6}{30}$

$r_c = \frac{6h}{30} = \frac{h}{5}$

$$V = \frac{1}{3}\pi\left(\frac{h}{5}\right)^2 h$$

$$V = \frac{1}{3}\pi\left(\frac{h^2}{25}\right)h$$

$$V = \frac{\pi h^3}{75}$$

$$\frac{dV}{dt} = \frac{3\pi}{75}h^2 \frac{dh}{dt}$$

$$\frac{\frac{dV}{dt}}{\frac{\pi}{25}h^2} = \frac{dh}{dt}$$

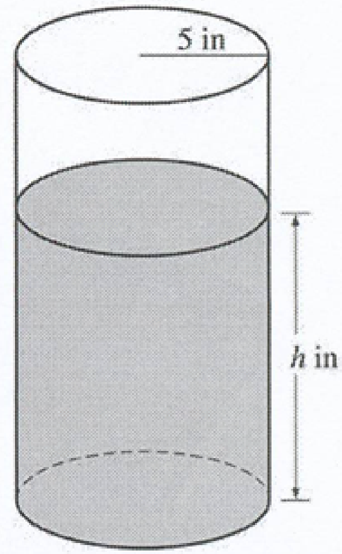
$$\frac{10\,cm^3/s}{\frac{\pi}{25}(20cm)^2} = dh/dt$$

$$.199\,cm/s = \frac{10\,cm^3/s}{50.2655\,cm^2} = dh/dt$$

AP® CALCULUS AB
2003 SCORING GUIDELINES

Question 5

A coffeepot has the shape of a cylinder with radius 5 inches, as shown in the figure above. Let h be the depth of the coffee in the pot, measured in inches, where h is a function of time t, measured in seconds. The volume V of coffee in the pot is changing at the rate of $-5\pi\sqrt{h}$ cubic inches per second. (The volume V of a cylinder with radius r and height h is $V = \pi r^2 h$.)



(a) Show that $\boxed{\dfrac{dh}{dt} = -\dfrac{\sqrt{h}}{5}}$.

(b) Given that $h = 17$ at time $t = 0$, solve the differential equation $\dfrac{dh}{dt} = -\dfrac{\sqrt{h}}{5}$ for h as a function of t.

(c) At what time t is the coffeepot empty?

$$\frac{dV}{dt} = -5\pi\sqrt{h} \text{ in}^3/\text{s} \qquad V = \pi r^2 h$$

$$V = \pi(5\,\text{in})^2 h$$

$$\boxed{V = 25\pi h}$$

$$\frac{dV}{dt} = 25\pi \frac{dh}{dt}$$

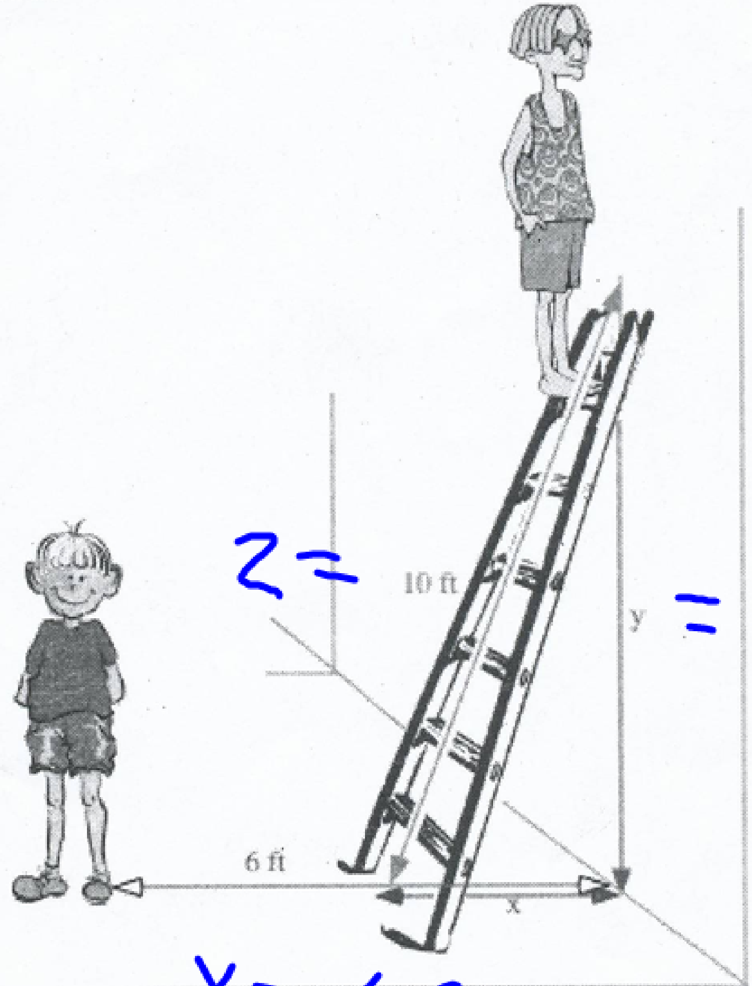
$$-\frac{\sqrt{5}}{5} \qquad \frac{-\cancel{5}\,\cancel{\pi}\sqrt{\pi}\ \text{in}^{\cancel{3}}/s}{\cancel{25}\,\cancel{\pi}\ \cancel{\text{in}^2}} = \cancel{25\pi\,\text{in}^2}\ \frac{dh}{dt}$$

Assignment
Calc 30 Text Page 290
#'s 9, 11, 12, 13, 14
AP Calc Text Page 239
#24

Pythagorean Problems

$$\boxed{x^2 + y^2 = r^2}$$

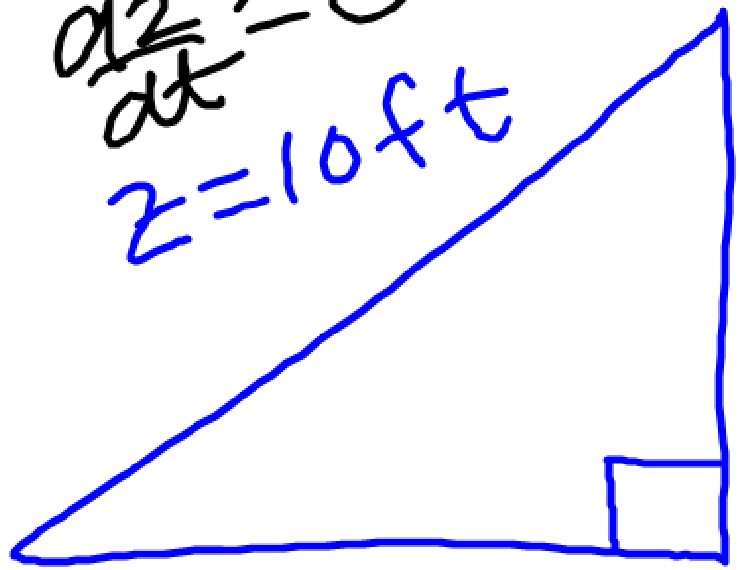
Joey is perched precariously the top of a 10-foot ladder leaning against the back wall of an apartment building (spying on an enemy of his) when it starts to slide down the wall at a rate of 4 ft per minute. Joey's accomplice, Lou, is standing on the ground 6 ft. away from the wall. How fast is the base of the ladder moving when it hits Lou?



$x = 6$ ft

$\frac{dz}{dt}=0$

$z=10\text{ ft}$



$\frac{dy}{dt}=-4\text{ ft/min}$

$y=8\text{ ft}$

$x=6\text{ ft}$

$\frac{dx}{dt}=?$

$x^2+y^2=z^2$

$(6)^2+y^2=(10)^2$

$y^2=100-36$

$y^2=64$

$y=\pm 8$

$$x^2 + y^2 = z^2$$

$$\cancel{2}x\frac{dx}{dt} + \cancel{2}y\frac{dy}{dt} = \cancel{2}z\frac{dz}{dt} = 0$$

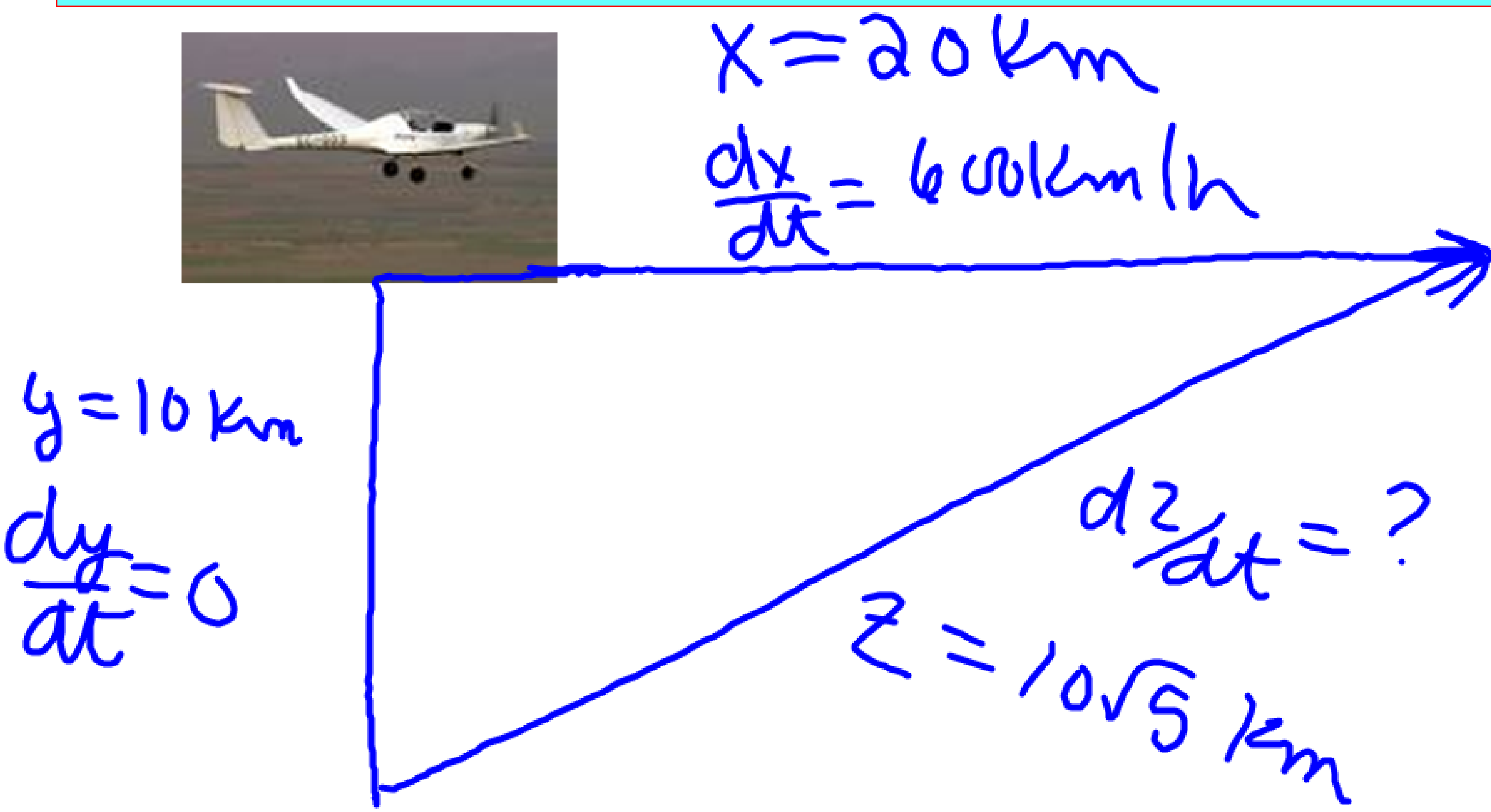
$$x\boxed{\frac{dx}{dt}} + y\frac{dy}{dt} = 0$$

$$\frac{dx}{dt} = \frac{-y}{x}\frac{dy}{dt}$$

$$\frac{dx}{dt} = -\frac{(8\cancel{ft})}{(6\cancel{ft})}(-4\,ft/min)$$

$$\frac{dx}{dt} = \frac{32}{6}\,ft/min$$

<u>Example</u> A plane flies horizontally with a speed of 600 km/h at an altitude of 10 km and passes directly over LeBoldus High School. Find the rate at which the distance from the plane to the school is changing when the horizontal distance of the plane is 20 km from the school. (Assume altitude doesn't change.)



$$x^2 + y^2 = z^2$$

$$(20)^2 + (10)^2 = z^2$$

$$\pm\sqrt{500} = z$$

$$+10\sqrt{5} = z$$

$$x^2 + y^2 = z^2$$

$$\cancel{2}x\frac{dx}{dt} + \cancelto{0}{\cancel{2}y\frac{dy}{dt}} = \cancel{2}z\frac{dz}{dt}$$

$$\frac{x}{z}\frac{dx}{dt} = \cancel{z}\boxed{\frac{dz}{dt}}$$

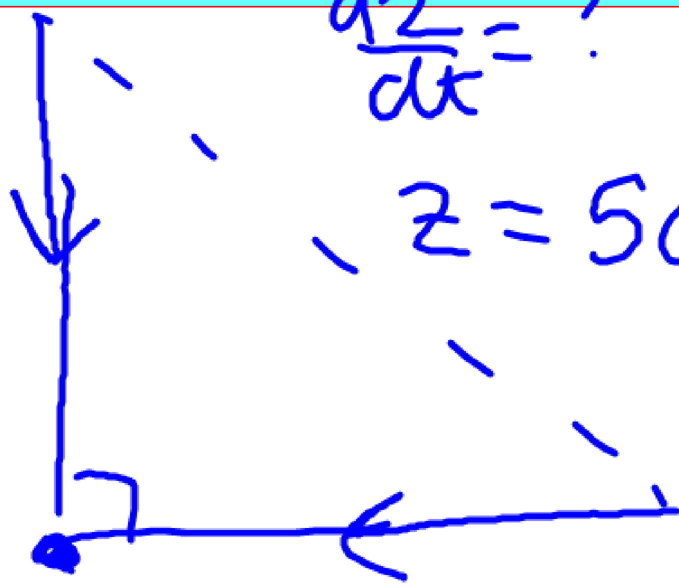
$$\left(\frac{20\,\cancel{km}}{10\sqrt{5}\,\cancel{km}}\right)\left(600\,km/h\right) = \frac{dz}{dt} = 536.656\ km/h$$

<u>Example</u> **A car approaches an intersection from the east at a speed of 12 m/s while a truck approaches from the north at a rate of 15 m/s. How fast is the distance between them decreasing when the car is 30 m east of the intersection and the truck is 40 m north of the intersection?**

$\frac{dz}{dt} = ?$

$\frac{dy}{dt} = -15 \text{ m/s}$

$y = 40 \text{m}$



$z = 50 \text{ m}$

$dx/dt = -12 \text{ m/s}$

$x = 30 \text{m}$

$$x^2 + y^2 = z^2$$

$$\frac{x\frac{dx}{dt} + y\frac{dy}{dt}}{z} = \cancel{z}\,\boxed{\frac{dz}{dt}}$$

$$\frac{(30\,m)(-12\,m/s) + (40\,m)(-15\,m/s)}{50\,m} = \frac{dz}{dt}$$

$$-19.2\,m/s = \frac{dz}{dt}$$

Example: A 13 foot ladder is leaning against the side of a building. If the foot of the ladder is pulled away from the building at a constant rate of 8 inches per second, how fast is the area of the triangle formed by the ladder, the building and the ground changing (in feet squared per second) at the instant when the top of the ladder is 12 feet above the ground?

$$\frac{dx}{dt} = 8 \text{ in/s} = .66 \text{ ft/s} \qquad \boxed{A = \frac{1}{2}xy}$$

$$x = 5 \text{ ft.}$$

$$A = \frac{1}{2} x y$$

$$\frac{dA}{dt} = \frac{1}{2}\left[x \frac{dy}{dt} + y \frac{dx}{dt} \right]$$

$$= \frac{1}{2}\left[5\,ft \left(\frac{-5}{18} \right) ft/s + (12\,ft)\left(\overset{2/3}{\cancel{.66}}\, ft/s \right) \right]$$

$$\frac{dA}{dt} = 3.266\ ft^2/s$$

$$\underline{3.306}$$

$$x^2 + y^2 = z^2$$

$$\cancel{2}x\frac{dx}{dt} + \cancel{2}y\frac{dy}{dt} = \cancel{2z\frac{dz}{dt}}^{0}$$

$$y\frac{dy}{dt} = -x\frac{dx}{dt}$$

$$\frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt} = -\frac{5}{12}(.66\,ft/s)$$

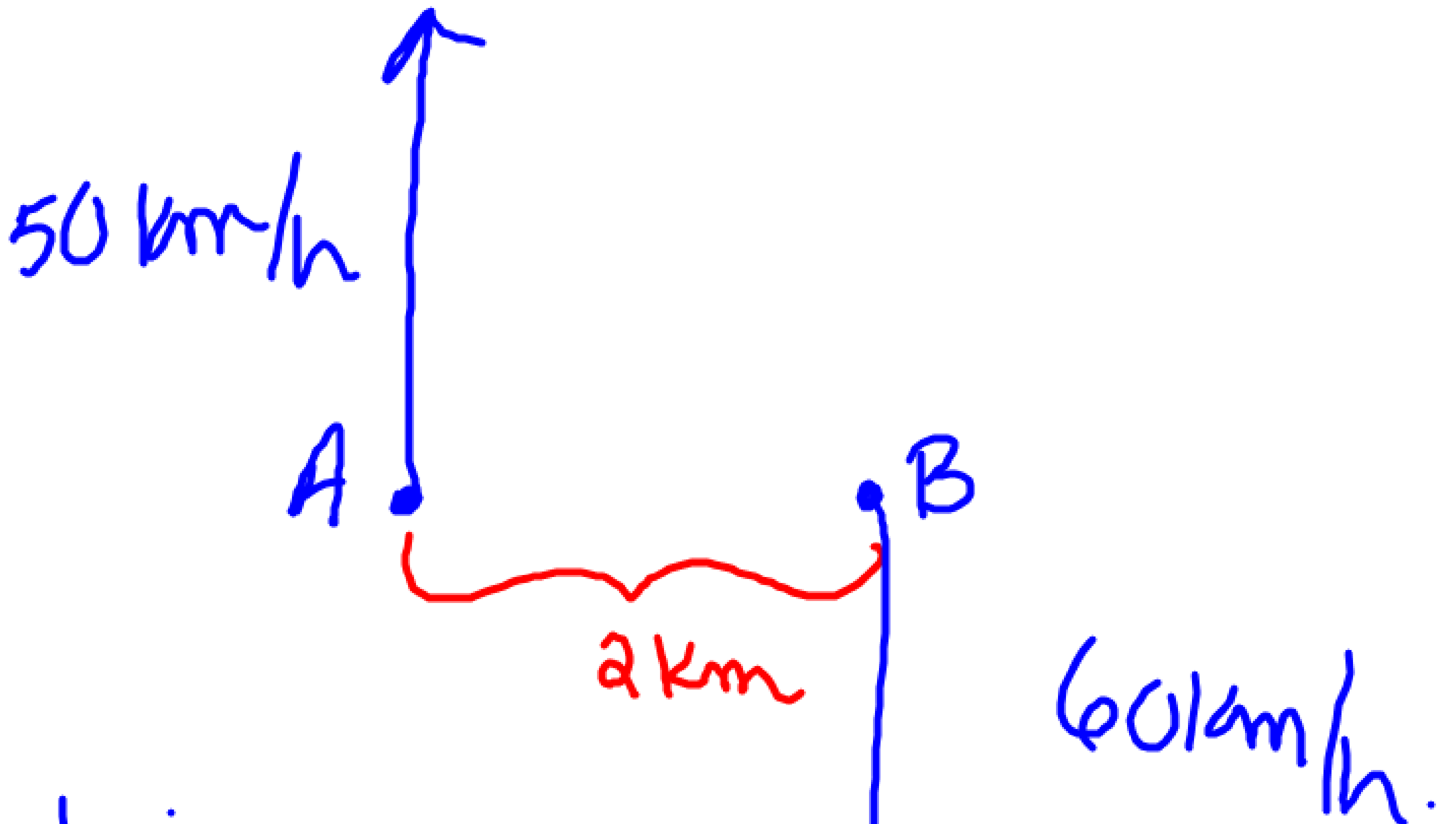
$$= -5/18\ ft/s$$

$$x^2 + y^2 = z^2$$

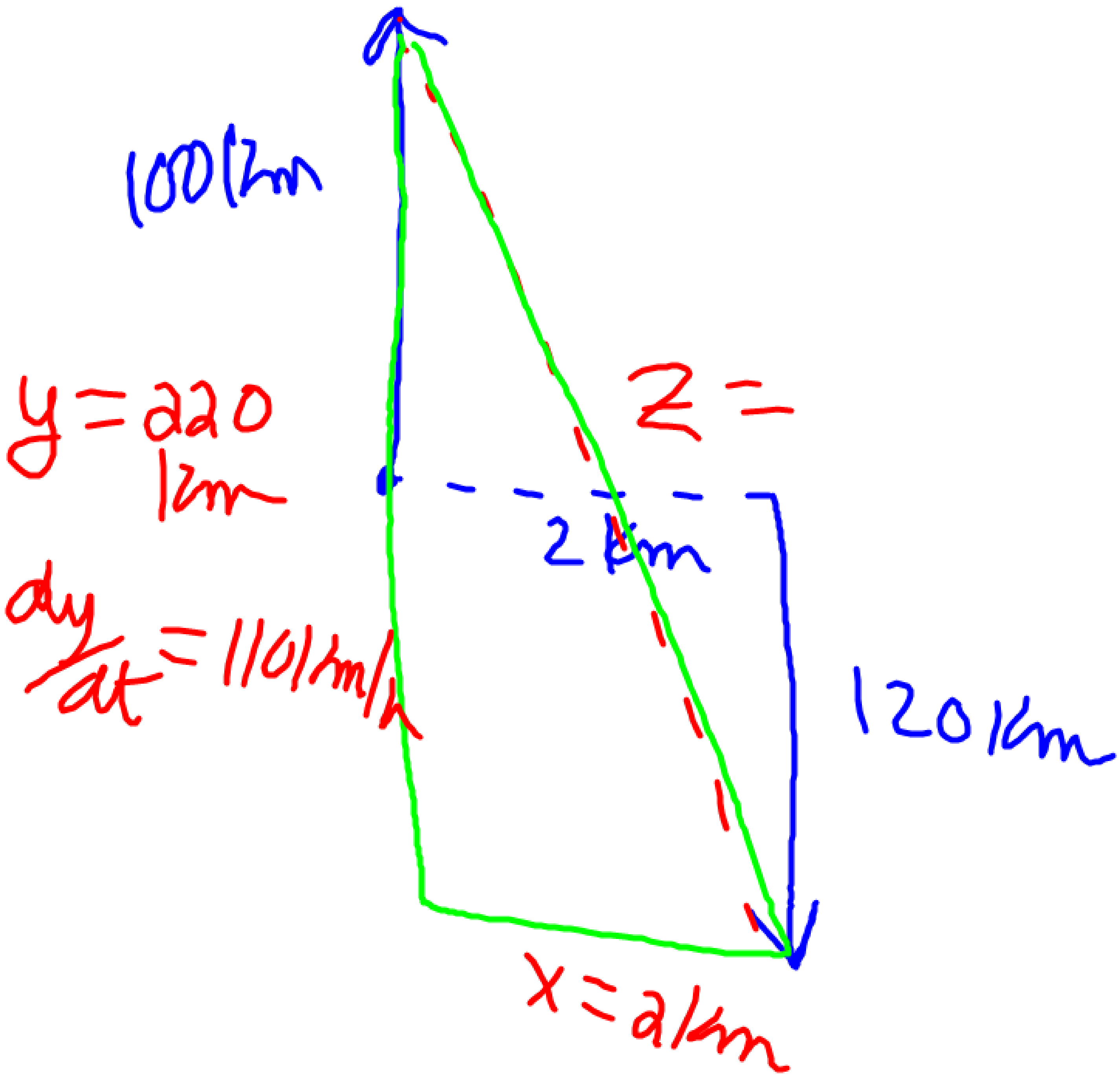
$$x^2 + (12)^2 = (13)^2$$

$$x^2 = 25$$

$$x = \pm 5$$



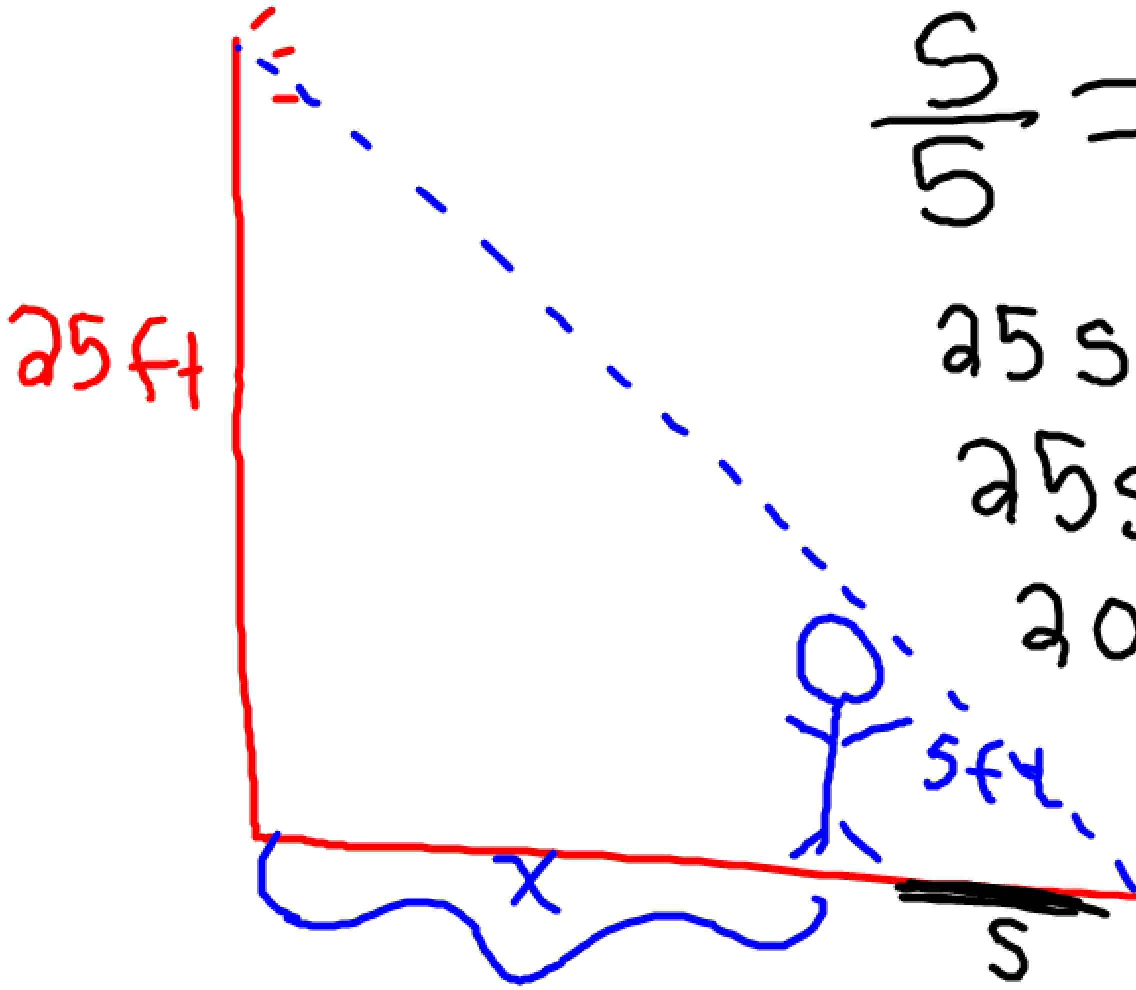
What is rate change between the Cars 2 hours into their trip?



1001km
y=220
1km
dy/dt=1101km/h
z=
2 km
120 km
x=21km

Shadow Problems

Example: A boy who is 5 feet tall is walking away from the base of a street light at a rate of 3 feet per second. If the streetlight is 25 feet high
a) at what rate is the boy's shadow changing?
b) at what speed is the boy's shadow moving?



$$\frac{dx}{dt} = 3\text{ ft/s}$$

$$\frac{s}{5} = \frac{x+s}{25}$$

$$25s = 5(x+s)$$

$$25s = 5x + 5s$$

$$20s = 5x$$

$$s = \frac{1}{4}x$$

$$\frac{ds}{dt} = \frac{1}{4}\frac{dx}{dt}$$

$$\frac{ds}{dt} = \frac{1}{4}\left(3\,\frac{ft}{s}\right) = \frac{3}{4}\,ft/s$$

b) Shadow moving at a speed of $3\,ft/s + \frac{3}{4}\,ft/s$

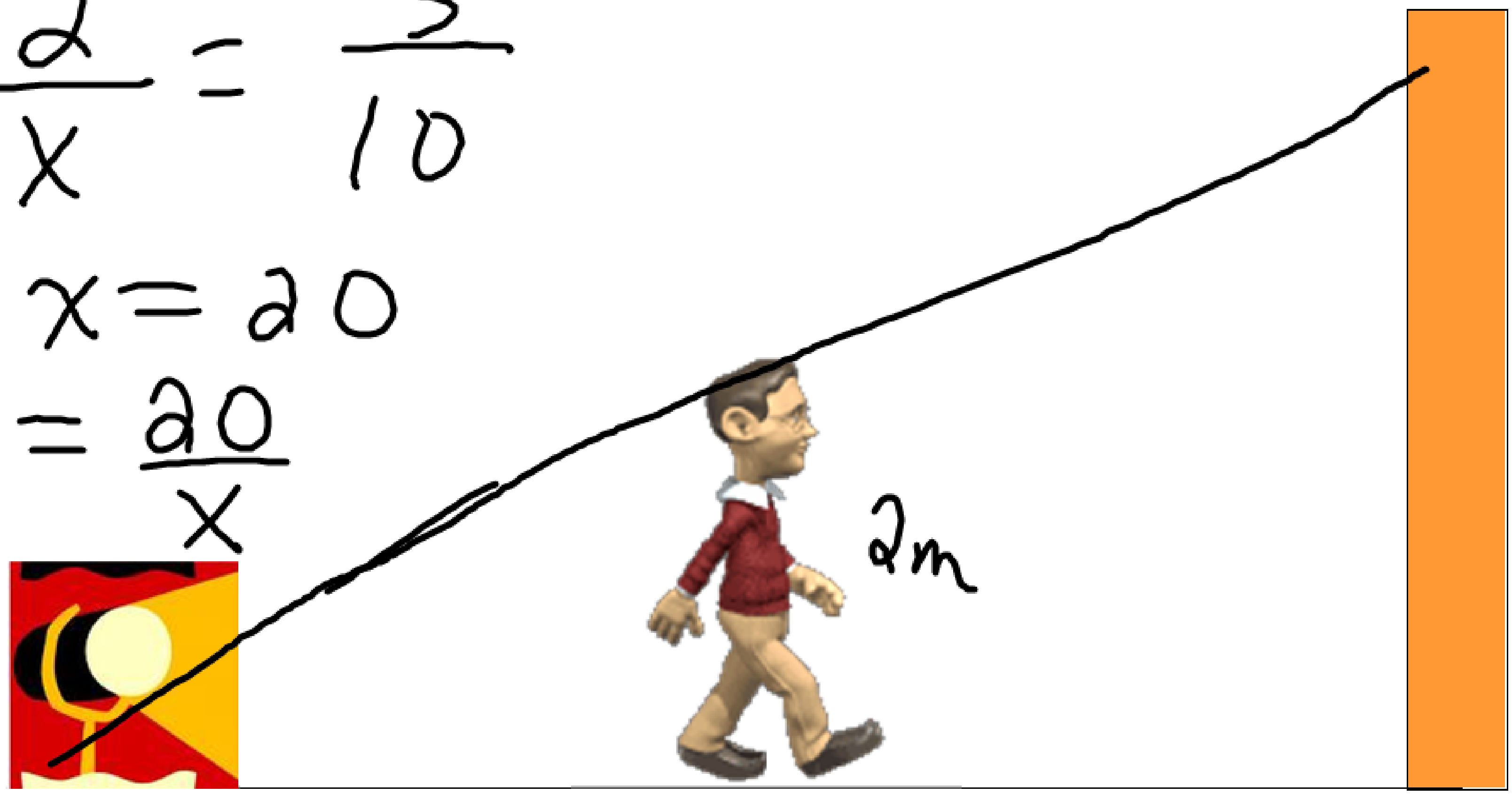
$3\,{}^{3}/_{4}\,ft/s$.

<u>Example</u> A spotlight on the ground shines on a wall 10 m away. A man 2 m tall walks from the spotlight to the wall at a speed of 1.2 m/s. How fast is the shadow on the wall decreasing when he is 3 m from the wall?

$$\frac{2}{x} = \frac{s}{10}$$

$$sx = 20$$

$$s = \frac{20}{x}$$



$x = 7$ 3m

$\frac{dx}{dt} = 1.2\,m/s$

$$s = 20x^{-1}$$

$$\frac{ds}{dt} = -20x^{-2}\frac{dx}{dt}$$

Shadow Problem Midnight Tutor

http://www.midnighttutor.com/related_rate_shadow.html

Assignment
Calc 30 Text Page 284
#'s 1, 2, 5, 8, 9, 10, 11, 12
AP Calc Text Page 240
29