4.5 L'Hopital's Rule

Let’s Review the Theorem:

If $\lim_{x \to c} \frac{f(x)}{g(x)}$ produces the indeterminate form of $\frac{0}{0}$ or $\frac{\infty}{\infty}$

and if $\lim_{x \to c} \frac{f'(x)}{g'(x)} = L,$ then $\lim_{x \to c} \frac{f(x)}{g(x)} = L.$

Example

$$\lim_{x\to 0} \frac{1-\cos(2x)}{5x}$$

$\lim_{x\to 0} 1-\cos(2x) = 0$

$\lim_{x\to 0} 5x = 0$

This limit produces the indeterminate form $\frac{0}{0}$.

L' Hospital's Rule Apply

$$\lim_{x\to 0} \frac{2\sin(2x)}{5} = \frac{2\sin(2\cdot 0)}{5} = \frac{0}{5} = 0$$

Should you Apply L'Hospital's Rule?

$$\cancel{\lim_{x\to 0}\frac{\sin x}{2x^2-1}}$$

$\lim_{x\to 0} \sin x = 0$

$\lim_{x\to 0} 2(0)^2 - 1 = -1$

NO

$$\cancel{\lim_{x\to 2}\frac{\sin x}{x^2-4}}$$

NO

$$\lim_{x\to 2}\frac{\sin(x-2)}{x^2-4}$$

$\lim_{x\to 2} \sin(2-2) = 0$

$\lim_{x\to 2} x^2 - 4 = 0$

Yes

Practice 2

$$\lim_{x \to 1} \frac{\ln(x)}{x^2 - 1}$$

$$\lim_{x \to 1} \ln(x) = 0$$

$$\lim_{x \to 1} x^2 - 1 = 0$$

This limit produces indeterminate form of $0/0$.

L'Hospital's Rule Applies

$$\lim_{x \to 1} \frac{\frac{1}{x}}{2x} = \frac{\left(\frac{1}{1}\right)}{2(1)} = \frac{1}{2}$$

Practice 3

x	5	3
$f(x)$	0	-3
$f'(x)$	7	4

The table above gives selected values of a twice-differentiable function $f(x)$.

Find $\lim_{x\to 3}\frac{f(2x-1)}{x^2-9}$

$$\lim_{x\to 3} f(2(3)-1) = 0$$

$$\lim_{x\to 3} x^2-9 = 0$$

This limit produces the indeterminate form of $\frac{0}{0}$.

$$\lim_{x\to 3}\frac{2f'(2x-1)}{2x} = \frac{2f'(5)}{6} = \frac{2(7)}{6} = \frac{7}{3}$$

Your Turn

Find $\lim_{x \to 0} \dfrac{e^x - \cos x}{4 \sin x}$.

$\lim_{x \to 0} e^x - \cos x = 0$

$\lim_{x \to 0} 4 \sin 0 = 0$

This limit produces the indeterminate form 0/0

L'Hospital's Applies

$$\lim_{x \to 0} \frac{e^x + \sin x}{4 \underline{\cos x}} = \frac{1 + 0}{4} = \frac{1}{4}$$

AP Question 1

$$\lim_{x \to 0} \frac{e^x - \cos x - 2x}{x^2 - 2x} =$$

$1 - 1 - 0 = 0$

$0 - 0 = 0$

$$\lim_{x \to 0} \frac{e^x + 5\cancel{\sin x} - 2}{2\cancel{x} - 2} = \frac{1-2}{-2} = \frac{1}{2}$$

(A) $-\frac{1}{2}$

(B) 0

(C) $\frac{1}{2}$

(D) 1

(E) nonexistent

AP Question 2

$$\lim_{x\to\infty}\frac{\ln\left(e^{3x}+x\right)}{x} = \qquad \frac{\infty}{\infty}$$

(A) 0

(B) 1

(C) 3

(D) ∞

$$\lim_{x\to\infty} \frac{\left(\frac{1}{e^{3x}+x}\cdot 3e^{3x}+1\right)}{1}$$

$$\lim_{x\to\infty} \frac{3e^{3x}+1}{e^{3x}+x}$$

$$\lim_{x\to\infty} \frac{9e^{3x}}{3e^{3x}+1} \rightarrow \lim_{x\to\infty} \frac{27\cancel{e^{3x}}}{9\cancel{e^{3x}}} = 3$$

2019 Exam

Free-Response Question 1

The functions f and h are twice-differentiable with $h(2) = 4$.

The function h satisfies $\boxed{h(x) = \dfrac{x^2 - 4}{1-(f(x))^3}}, x \neq 2$. It is known that

$\lim_{x\to 2} h(x)$ can be evaluated using L'Hospital's Rule.

Use $\lim_{x\to 2} h(x)$ to find $f(2)$ and $f'(2)$. Show the work that leads to your answers.

Since $h(2) = 4$

h diff $\rightarrow$ h cont.

$\therefore \lim_{x\to 2} h(x) = 4$

$$\lim_{x\to 2} 1-(f(x))^3 = 0$$

$$1-(f(2))^3 = 0$$

$$1 = (f(2))^3$$

$$1 = f(2) \checkmark$$

$$\lim_{x\to 2} \frac{2x}{-3(f(x))^2 \cdot f'(x)} = 4$$

$$\frac{2(2)}{-3(f(2))^2 \cdot f'(2)} = 4$$

$$\frac{4}{-3(f'(2))} = 4$$

$$4 = 4(-3)f'(2)$$

$$-\frac{1}{3} = f'(2)$$

L'Hospital's Rule Assignment

1. Let f be the function defined by $f(x) = 3x + 2e^{-3x}$, and let g be a differentiable function with derivative given by $g'(x) = 4 + \frac{1}{x}$. It is known that $\lim_{x\to\infty} g(x) = \infty$. The value of $\lim_{x\to\infty} \frac{f(x)}{g(x)}$ is

2. $\lim_{t\to 0} \frac{\sin t}{\ln(2e^t - 1)} =$

3. $\lim_{x\to \frac{\pi}{2}} \frac{3\cos x}{2x - \pi}$ is

4. Let f be the function defined by $f(x) = 2x + 3e^{-5x}$, and let g be a differentiable function with derivative given by $g'(x) = \frac{1}{x} + 4\cos\left(\frac{5}{x}\right)$. It is known that $\lim_{x\to\infty} g(x) = \infty$. The value of $\lim_{x\to\infty} \frac{f(x)}{g(x)}$ is

5. $\lim_{x\to 0} \frac{6e^{4x} - 2e^{3x} - 4}{\sin(2x)} =$

6. Particle P moves along the y-axis so that its position at time t is given by $y(t) = 4t - \frac{2}{3}$ for all times t. A second particle, particle Q, moves along the x-axis so that its position at time t is given by $x(t) = \frac{\sin(\pi t)}{2-t}$ for all times $t \neq 2$.

a) As time t approaches 2, what is the limit of the position of particle Q ? Show the work that leads to your answer.

b) Show that the velocity of particle Q is given by $v_Q(t) = \frac{2\pi\cos(\pi t) - \pi t\cos(\pi t) + \sin(\pi t)}{(2-t)^2}$ for all times $t \neq 2$.

c) Find the rate of change of the distance between particle P and particle Q at time $t = \frac{1}{2}$. Show the work that leads to your answer.

Answers

1. 3/4 2. 1/2 3. -3/2 4. 1/2 5. 9 6.a) -π b) see answer key c) $\frac{76}{9\sqrt{5}}$