4.4 Optimization Problems

Max/Min Problems

These problems often referred to as **max min** problems.

There are many real world applications of these types of problems.

Number Problems

<u>Ex.1</u> **Find two non-negative numbers whose sum is 15 such that the product of one with the square of the other is a maximum.**

let $x = 1^{st}$ non neg # ①

$y = 2^{nd}$ " " "

$\boxed{x + y = 15}$ ①

$\boxed{P = xy^2}$ * deriv.

$x = 15 - y$

$P = (15 - y)y^2$ ①

$P = 15y^2 - y^3$

$P' = 30y - 3y^2$ 1

$30y - 3y^2 = 0$

$3y(10 - y) = 0$

$3y = 0$ or 1 $y = 10$

$y = 0$

Proof:

$P'' = 30 - 6y$ 1

$P''(0) \cancel{= 30} > 0$

$P''(10) = 30 - 6(10) < 0$

$\therefore y = 10$ max

Find x

$x + y = 15$

$x + 10 = 15 \rightarrow x = 5$ 1

The 2 non-negative #s are 5 and 10.

1

<u>Ex.2</u> **Two numbers have a product of 16. Find these numbers if the sum of their squares is to be a minimum. What is the minimum sum of the squares?**

let $x = 1^{st}$ # $y = 2^{nd}$ #

$$\boxed{xy = 16}$$

$$y = \frac{16}{x}$$

$$S = x^2 + y^2$$

$$S = x^2 + \left(\frac{16}{x}\right)^2$$

$$S = x^2 + 256x^{-2}$$

$$S' = 2x - \frac{512}{x^3}$$

$$s' = 0$$

$$2x - \frac{512}{x^3} = 0$$

$$2x = \frac{512}{x^3}$$

$$2x^4 = 512$$

$$x^4 = 256$$

$$x = \pm 4$$

$$\cancel{\begin{array}{c} S' \ \infty \\ \hline x = 0 \end{array}}$$

Proof

$$S'' = 2 + \frac{1536}{x^4}$$

$$S''(4) = 2 + \frac{1536}{(4)^4} > 0$$

$x = 4$ min

$x = -4$ min

Find y

$xy = 16$

$4y = 16$

$y = 4$

$x = 4$

if $x = -4$

$-4y = 16$

$y = -4$

$S = x^2 + y^2$

$(4, +4)$

$(-4, -4)$

$S = (4)^2 + (4)^2$

$S = 32$

Area Perimeter Problems

<u>Ex.3</u> A farmer has 1000m of fencing and wants to enclose a rectangular pasture bordering a river. Find the dimensions that will maximize the area. What is the maximum area?

let L = length W = width

$$\boxed{2w + L = 1000} \qquad \boxed{A = LW} \ast$$

$$L = 1000 - 2w$$

$$A = (1000 - 2w)w$$

$$A = 1000w - 2w^2$$

$$A' = 1000 - 4w$$

$$\underline{A' = 0}$$

$$1000 - 4w = 0$$

$$1000 = 4w$$

$$250 = w$$

Proof

$A'' = -4$

CD always

$\therefore w = 250$ max.

Find L

$L = 1000 - 2(250)$

$L = 500\text{ m}$

$A = (500)(250)$

$A = 125000\text{ m}^2$

The dimensions are 250 m X 500 m and max area is 125000 m².

82. If $y = 2x - 8$, what is the minimum value of the product xy ?

(A) -16 (B) -8 (C) -4 (D) 0 (E) 2

$P = xy$

$P = x(2x-8)$

$P = 2x^2 - 8x$

$P' = 4x - 8$

$4x - 8 = 0$

$\boxed{x = 2}$

$y = 4$

Assignment

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#'s 2,3,5,7,13,14

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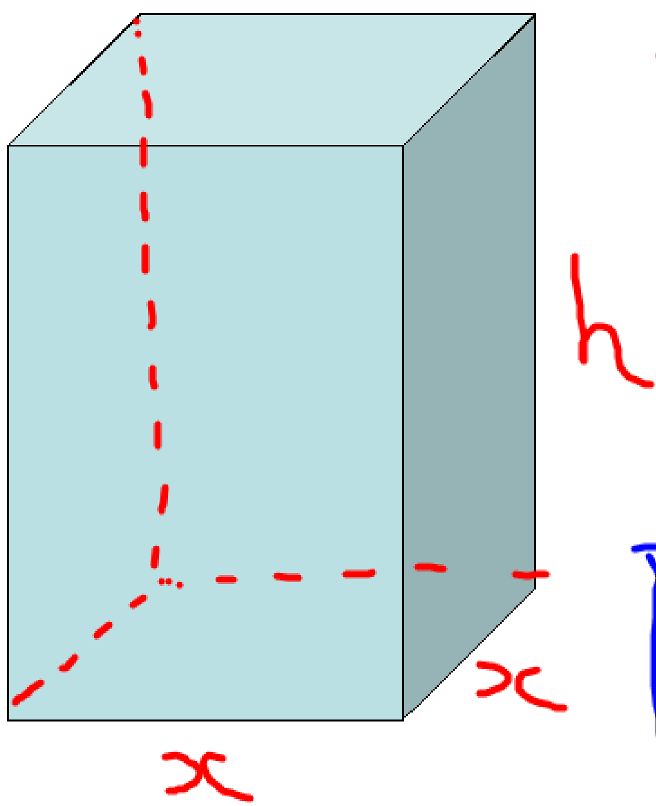
Volume Surface Area Problems

<u>Ex.4</u> If 200cm^2 of material is available to make a box with a square base and an open top. Find the largest possible volume of the box. What dimensions maximize the volume of the box?



let x = width/length base

h = height

$$\boxed{V = x^2 h} \ *$$

$$\boxed{200 = x^2 + 4xh}$$

$$\frac{200 - x^2}{4x} = h$$

$$V = x\left(\frac{200 - x^2}{4}\right)$$

$$V = 50x - \frac{x^3}{4}$$

$$V' = 50 - \frac{3}{4}x^2$$

$$\underline{V' = 0}$$

$$50 - \frac{3}{4}x^2 = 0$$

$$50 = \frac{3}{4}x^2$$

$$\frac{200}{3} = x^2$$

$$\pm\sqrt{\frac{200}{3}} = x$$

$$\sqrt{\frac{200}{3}} = x$$

Proof

$$V'' = -\frac{6}{4}x$$

$$V''\left(\sqrt{\frac{200}{3}}\right) = -\frac{6}{4}\left(\sqrt{\frac{200}{3}}\right) < 0$$

$\therefore$ CD $\therefore$ $x = \sqrt{\frac{200}{3}}$ max

$$h = \frac{200 - x^2}{4x} = \frac{200 - \frac{200}{3}}{4\left(\sqrt{\frac{200}{3}}\right)} = 4.08\text{ cm}$$

Dimensions of the box are

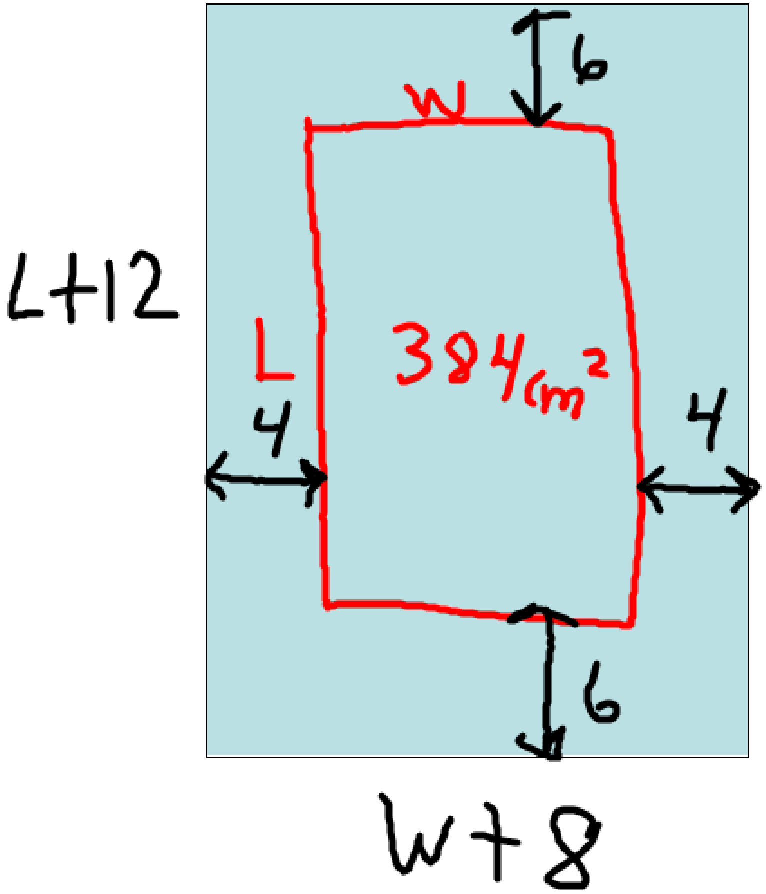
$8.16\,cm \times 8.16\,cm \times 4.08\,cm$

and the volume is $V = (8.16\,cm)^2(4.08\,cm)$

$V = 271.67\,cm^3$.

Poster Problems

<u>Ex.5</u> The top and bottom margins of a poster are six cm and the side margins are 4 cm. If the printed area on the poster is 384 cm^2, find the dimensions of the poster with the smallest area.



let w = width print

L = length print

$$\boxed{WL = 384} \qquad L = \frac{384}{w}$$

$$\boxed{A = (L+12)(w+8)} \; *$$

$$A = WL + 8L + 12W + 96$$

$$A = 384 + 8L + 12W + 96$$

$$A = 480 + 8L + 12W$$

$$A = 480 + 8\left(\frac{384}{W}\right) + 12W$$

$$A = 480 + 3072W^{-1} + 12W$$

$$A' = -\frac{3072}{W^2} + 12$$

$A' = 0$

$0 = -\frac{3072}{w^2} + 12$

$\frac{3072}{w^2} = 12$

$3072 = 12w^2$

$256 = w^2$

$\pm 16 = w$

$w = 16$

$\cancel{\frac{A' \infty}{w \neq 0}}$

Proof

$A'' = \frac{6144}{w^3}$

$A''(16) = \frac{6144}{(16)^3} > 0 \therefore CU$

$w = 16$ is a min.

Find L

$$L = \frac{384}{16} = 24$$

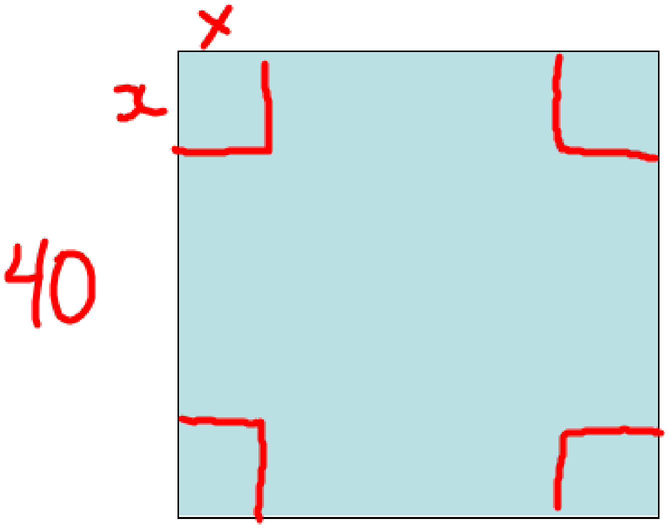
Dimensions of the entire poster is 36cm x 24cm.

* 1 equation

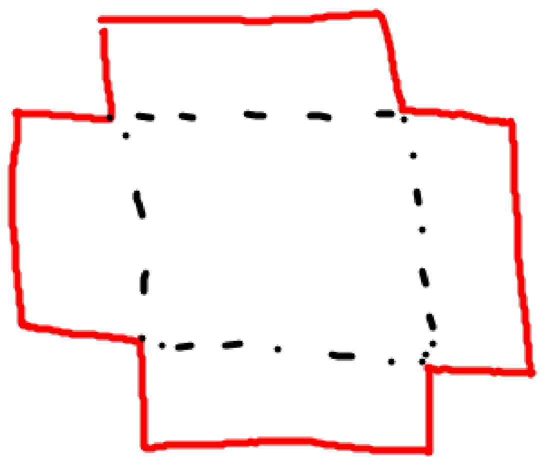
Volume Problems

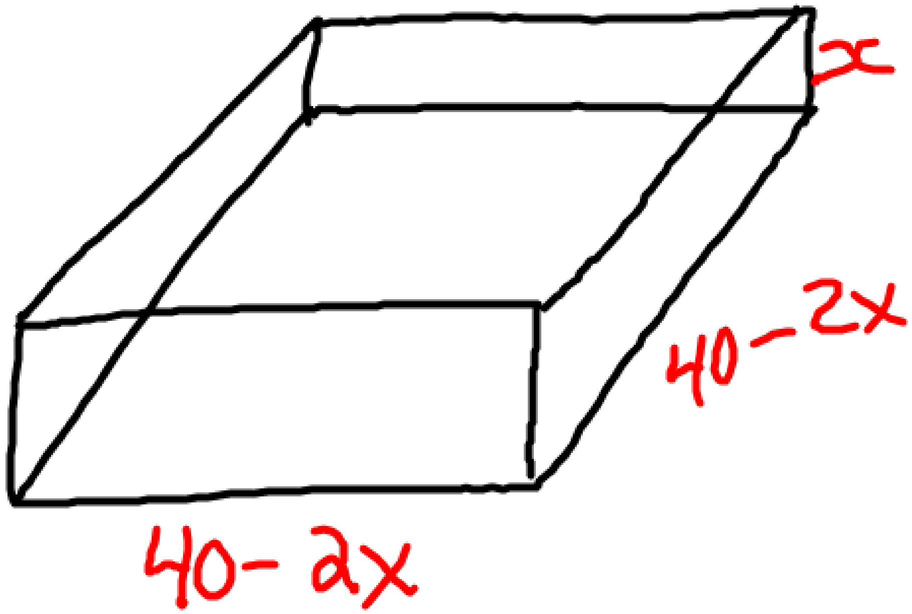
http://www.youtube.com/watch?v=3GYv-BZYYdg&feature=related

<u>Ex.6</u> **A box is to be made from a square piece of cardboard by cutting a square out of each corner and turning the sides up to form walls. Given that the cardboard is 40cm by 40 cm find the dimensions of the box that will maximize the volume.**



let x = length of cut





$$V = (40-2x)^2 x$$

$$V = (1600 - 160x + 4x^2)x$$

$$V = 4x^3 - 160x^2 + 1600x$$

$V' = 12x^2 - 320x + 1600$

using quad formula

$\cancel{x = 20}$ or $x = 20/3$

Proof

$V'' = 24x - 320$

$V''\left(\frac{20}{3}\right) = 24\left(\frac{20}{3}\right) - 320 < 0 \therefore CD$

$\therefore x = 20/3$ max.

height $= \frac{20}{3}$ cm $= 6.67$cm

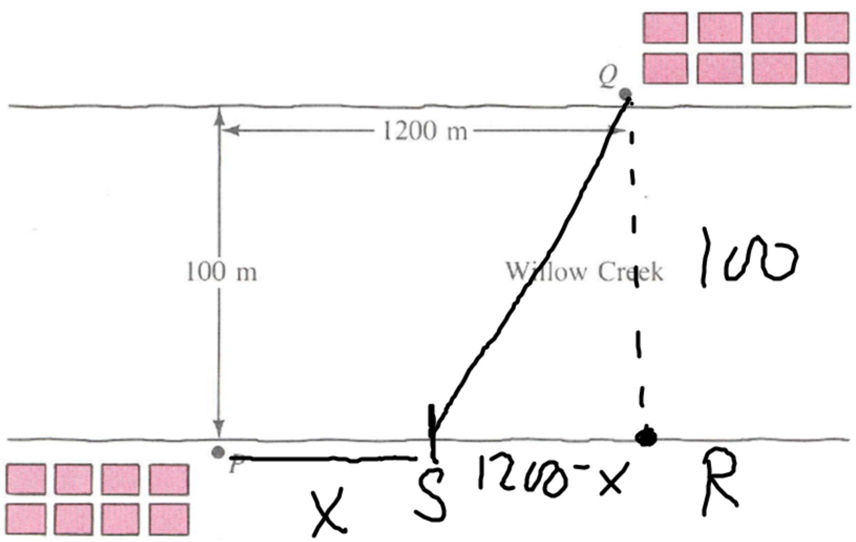
Width/Length $= 40 - 2\left(\frac{20}{3}\right) = 26.67$

The dimensions of box are

26.67cm x 26.67cm x 6.67cm.

Real Life Application

A cable television company is laying cable in an area with underground utilities. Two subdivisions are located on opposite sides of Willow Creek, which is 100 m wide. The company has to connect points P and Q with cable, where Q is on the north bank 1200 m east of P. It costs \$40/m to lay cable underground and \$80/m to lay cable underwater. What is the least expensive way to lay the cable?



$C(0) = 96333$

$C(1200) = 56000$

$C(1142) = 54928$

$$SQ = \sqrt{(100)^2 + (1200 - x)^2}$$

$$Cost = 40x + 80\sqrt{(100)^2 + (1200 - x)^2}$$

$$Cost = 40x + 80\sqrt{10000 + 1440000 - 2400x + x^2}$$

$$Cost = 40x + 80\sqrt{1450000 - 2400x + x^2}$$

$$C' = 40 + 80\,\tfrac{1}{2}(1450000 - 2400x + x^2)^{-1/2} \cdot (-2400 + 2x)$$

$$C' = 40 + \frac{\cancel{80}\,40(2x-2400)}{\cancel{2}\sqrt{1450000-2400x+x^2}}$$

$$C' = 0$$

$$-40 = \frac{40(2x-2400)}{\sqrt{1450000-2400x+x^2}}$$

$$-1 = \frac{2x-2400}{\sqrt{1450000-2400x+x^2}}$$

$$\left(-\sqrt{1450000-2400x+x^2}\right)^2=(2x-2400)^2$$

$$1450000-2400x+x^2 = 4x^2-9600x+5760000$$

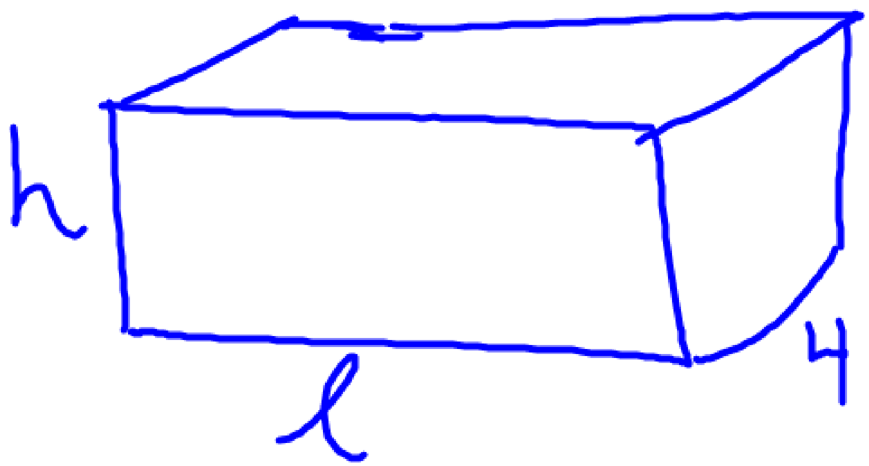
$$0 = 3x^2-7200x+4310000$$

using quad formula

$x \simeq 1258$ or $x \cong 1142$

1982 AP Exam Question

A tank with a rectangular base and rectangular sides is to be open at the top. It is to be constructed so that its width is 4 metres and its volume is 36 cubic metres. If building the tank costs $10 per square metre for the base and $5 per square metre for the sides, what is the cost of the least expensive tank?



let l = length
h = height

$$\boxed{36 = 4hl} \rightarrow l = \frac{36}{4h} = \frac{9}{h}$$

$$C = 10(4l) + 5(2hl) + 5(2 \cdot 4h)$$

$$\boxed{C = 40l + 10hl + 40h}$$

$$C = 40\left(\frac{9}{h}\right) + 10h\left(\frac{9}{h}\right) + 40h$$

$$C = 360h^{-1} + 90 + 40h$$

$$C' = -360h^{-2} + 40$$

$$C' = -\frac{360}{h^2} + 40$$

$$\cancel{\frac{C' \, \infty}{h = 0}}$$

$C' = 0$

$$-\frac{360}{h^2} + 40 = 0$$

$$40 = \frac{360}{h^2}$$

$$40h^2 = 360$$

$$h^2 = 9$$

$$h = \pm 3$$

$$h = 3$$

Proof

$C'' = \frac{720}{h^3}$

$C''(3) = \frac{720}{(3)^3} > 0$

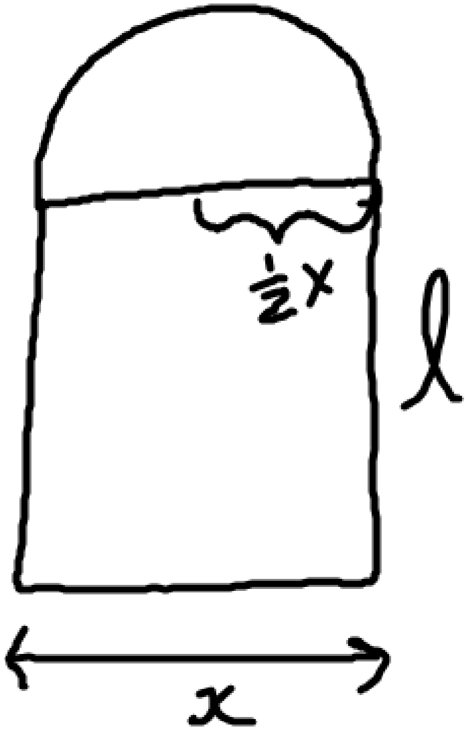
$\therefore$ CU

$\therefore h = 3 \text{ min}$

$C = \frac{360}{3} + 90 + 40 \cdot 3$

$= 120 + 90 + 120$

$= \$330$ min cost.



P = 16 ft

Max Area

Let l = length rectangle
x = width rectangle

$$A = xl + \frac{1}{2}\pi\left(\frac{x}{2}\right)^2$$

$$A = xl + \frac{\pi x^2}{8}$$

$$\boxed{16 = x + 2l + \frac{x\pi}{2}}$$

$$16 - x - \frac{\pi}{2}x = 2l$$

$$8 - \frac{x}{2} - \frac{\pi x}{4} = l$$

$$A = x\left(8 - \frac{x}{2} - \frac{\pi x}{4}\right) + \frac{\pi x^2}{8}$$

$$A = 8x - \frac{x^2}{2} - \frac{\pi x^2}{4} + \frac{\pi x^2}{8}$$

$$A' = 8 - x - \frac{2\pi x}{4} + \frac{\pi x}{4}$$

$$A' = 8 - x - \frac{\pi}{4}x$$

$$0 = 8 - x - \frac{\pi}{4}x$$

$$x + \frac{\pi}{4}x = 8$$

$$x\left(1 + \pi/4\right) = 8$$

$$x = \frac{8}{1 + \pi/4}$$

$$x \simeq 4.48$$

<u>Prob</u>

$$A'' = -1 - \pi/4$$

$\therefore$ CD

$\therefore$ 4.48 rel max

Assignment

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