4.4 Integrals of Inverse Trigonometric Functions

Recently we learned how to differentiate inverse trigonometric functions. We also need to know how to integrate inverse trigonometric functions.

Video Introducing Inverse Trigonometric Integrals

$$\int \frac{du}{\sqrt{a^2 - u^2}} = sin^{-1}\left(\frac{u}{a}\right) + C$$

$$\int \frac{du}{a^2 + u^2} = \frac{1}{a} tan^{-1}\left(\frac{u}{a}\right) + C$$

$$\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a} sec^{-1}\left(\frac{u}{a}\right) + C$$

Example 1

$u^2 = x^2$

$u = x$

$du = dx$

$a^2 = 16$

$a = 4$

$$\int \frac{dx}{\sqrt{16 - x^2}}$$

$$\int \frac{du}{\sqrt{a^2 - u^2}}$$

$\sin^{-1}\left(\frac{u}{a}\right) + C$

$\sin^{-1}\left(\frac{x}{4}\right) + C$

Example 2

$$\int \frac{3}{25+x^2}\,dx$$

$$3\int \frac{du}{a^2+x^2}$$

$u^2 = x^2$

$u = x$

$du = dx$

$a^2 = 25$

$a = 5$

$$3\left[\frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right)+C\right]$$

$$= 3\left[\frac{1}{5}\tan^{-1}\left(\frac{x}{5}\right)+C\right]$$

$$= \frac{3}{5}\tan^{-1}\left(\frac{x}{5}\right)+C$$

Example 3

$u^2 = 4x^2$ $\qquad a^2 = 1$

$u = 2x \qquad \frac{u}{2} = x \qquad a = 1$

$$\int \frac{8}{x\sqrt{4x^2-1}}\,dx$$

$du = 2\,dx$

$\frac{1}{2}du = dx$

$$8\int \frac{1}{x\sqrt{4x^2-1}}\,dx$$

$$8\int \frac{\frac{1}{\cancel{2}}du}{\frac{u}{\cancel{2}}\sqrt{u^2-a^2}} = 8\int \frac{du}{u\sqrt{u^2-a^2}} = 8\left[\frac{1}{a}\sec^{-1}\left(\frac{u}{a}\right)+C\right]$$

$$= 8\sec^{-1}(2x)+C$$

Example 4

$$\int \frac{x}{x^4 + 36}\,dx$$

$u^2 = x^4$

$u = x^2$

$du = 2x\,dx$

$\frac{1}{2}du = x\,dx$

$a^2 = 36$

$a = 6$

$$\frac{1}{2}\int \frac{du}{u^2 + a^2} = \frac{1}{2}\left[\frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C\right]$$

$$= \frac{1}{2}\cdot\frac{1}{6}\tan^{-1}\left(\frac{x^2}{6}\right) + C$$

$$= \frac{1}{12}\tan^{-1}\left(\frac{x^2}{6}\right) + C$$

$$\int \frac{1}{x^2+4x+5}\,dx =$$

Sometimes we need to complete the square to integrate!

$$\int \frac{1}{x^2+4x+4-4+5}$$

$$u^2 = (x+2)^2 \qquad a^2 = 1$$

$$u = x+2 \qquad a = 1$$

$$du = dx$$

$$\int \frac{1}{(x+2)^2+1}$$

$$\int \frac{1}{u^2+a^2}\,du = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right)+C$$

$$= \tan^{-1}(x+2)+C$$

Your Turn

$$\int \frac{1}{5x^2 - 30x + 65}\,dx$$

$$\int \frac{1}{5(x^2-6x+13)}\,dx$$

$$u^2 = (x-3)^2 \qquad a^2 = 4$$

$$a = 2$$

$$u = x-3$$

$$du = dx$$

$$\frac{1}{5}\int \frac{dx}{x^2-6x+9-9+13}$$

$$\frac{1}{5}\int \frac{dx}{(x-3)^2+4}$$

$$= \frac{1}{5}\int \frac{du}{u^2+a^2}$$

$$= \frac{1}{5}\left[\frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right)+C\right]$$

$$= \frac{1}{5}\left[\frac{1}{2}\tan^{-1}\left(\frac{x-3}{2}\right)+C\right]$$

$$= \frac{1}{10}\tan^{-1}\left(\frac{x-3}{2}\right)+C$$

$$\int \frac{x^3}{x^2+1}\,dx$$

Sometimes we need to use long division before we can integrate!

$$\begin{array}{r} x \\ x^2+1 \overline{\smash{\big)}\, x^3 + 0x} \\ -(x^3 + x) \\ \hline -x \end{array}$$

$$\int \left(x - \frac{x}{x^2+1} \right) dx$$

$$= \int x\,dx - \int \frac{x\,dx}{x^2+1}$$

u sub

let $u = x^2+1$

$du = 2x\,dx$

$\frac{1}{2}du = x\,dx$

$$= \frac{x^2}{2} - \frac{1}{2}\int \frac{1}{u}\,du$$

$$= \frac{x^2}{2} - \frac{1}{2}\ln|u| + C$$

$$= \frac{x^2}{2} - \frac{1}{2}\ln|x^2+1| + C$$

$$\int \frac{x^3+3x^2-1}{x^2+1}\,dx$$

$$\begin{array}{r|l} & x+3 \\ \hline x^2+1 & x^3+3x^2+0x-1 \\ & -(x^3+0x^2+x) \\ \hline & 3x^2-x-1 \\ & -(3x^2+0x+3) \\ \hline & -x-4 \end{array}$$

$$\int (x+3)\,dx - \int \frac{x+4}{x^2+1}\,dx$$

split up

$$= \frac{x^2}{2}+3x-\left[\int \frac{x}{x^2+1}\,dx + \int \frac{4}{x^2+1}\,dx\right]$$

let $u = x^2+1$

$du = 2x\,dx$

$\frac{1}{2}du = x\,dx$

$$= \frac{x^2}{2} + 3x - \left[\frac{1}{2} \int \frac{du}{u} + 4 \int \frac{dx}{x^2+1} \right.$$

arctan

$$= \frac{x^2}{2} + 3x - \left[\frac{1}{2} \ln|u| + 4 \tan^{-1}(x) \right] + C$$

$$= \frac{x^2}{2} + 3x - \frac{1}{2} \ln|x^2+1| - 4 \tan^{-1} x + C$$

$$\int_1^2 \frac{x^2 - x - 5}{x+2}$$

$$\begin{array}{r} x - 3 \\ x+2 \overline{\smash{\big)}\, x^2 - x - 5} \\ -(x^2 + 2x) \\ \hline -3x - 5 \\ -(-3x - 6) \\ \hline 1 \end{array}$$

$$\int_1^2 (x-3)\,dx + \int_1^2 \frac{1}{x+2}\,dx$$

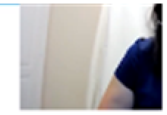
$$= \frac{x^2}{2} - 3x \Big|_1^2 + \ln|x+2| \Big|_1^2$$

$$= \left(\frac{(2)^2}{2} - 3(2)\right) - \left(\frac{1}{2} - 3\right) + \ln 4 - \ln 3$$

$$= 2 - 6 - \frac{1}{2} + 3 + \ln(4/3)$$

$$= -\frac{3}{2} + \ln(4/3)$$

Integration of Rational Functions



Given a rational function to integrate:

1. If a recognized antiderivative, such as the antiderivative of an inverse trig function or logarithmic function appears, then find the antiderivative.

2. If the degree of the numerator $\geq$ to the degree of the denominator, use long division to rewrite the integrand. Lastly, find the antiderivative.

Assignment First Handout #'s
1-8
Second Handout #'s
1,2,4,7,8,10,12,16