4.3 Derivatives of Inverse Trigonometric Functions

$y = \sin^{-1}(x)$ is known as the inverse of

$y = \sin x.$

For this to be true, $x = \sin y$ is the same as $y = \sin^{-1}(x).$

Lets try to develop a formula to find the derivative of arcsin.

AP Classroom

$$y = \sin^{-1}(x)$$

take sine of both sides

$$\sin y = \sin(\sin^{-1} x)$$

$$\sin y = x$$

$$\cos y \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\cos y}$$

$$\boxed{\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}}$$

$$\sin y = \frac{x}{1}$$

$$x^2 + a^2 = 1$$

$$a^2 = 1 - x^2$$

$$a = \sqrt{1-x^2}$$

$$\cos y = \frac{\sqrt{1-x^2}}{1}$$

Therefore we have:

$$\frac{\mathrm{d}}{\mathrm{dx}}\sin^{-1} u = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$$

Now try to derive the derivative of y=arccos x

$$y = \cos^{-1} x$$

$$\cos y = \cos(\cos^{-1} x)$$

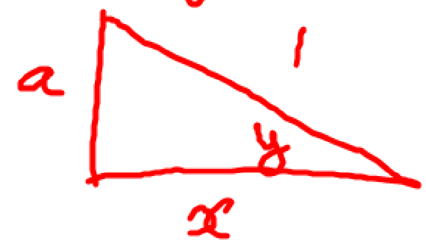
$$\cos y = x$$

$$-\sin y \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{-1}{\sin y}$$

$$\boxed{\frac{dy}{dx} = \frac{-1}{\sqrt{1-x^2}}}$$

$$\cos y = x$$



$$a^2 + x^2 = 1$$

$$a^2 = 1 - x^2$$

$$a = \sqrt{1-x^2}$$

$$\therefore \sin y = \frac{\sqrt{1-x^2}}{1}$$

$$\frac{dy}{dx}\cos^{-1}u = \frac{-1}{\sqrt{1-u^2}}\frac{du}{dx}$$

Now try to derive the derivative of y=arctan x

$$y = \tan^{-1} x$$

$$\tan y = \tan(\tan^{-1} x)$$

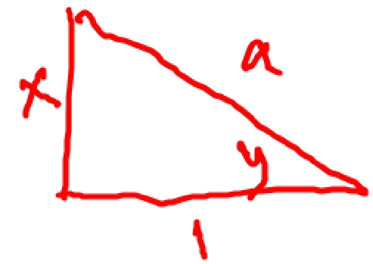
$$\tan y = x$$

$$\sec^2 y \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\sec^2 y}$$

$$\frac{dy}{dx} = \frac{1}{\left(\sqrt{x^2+1^2}\right)^2}$$

$$\tan y = x$$



$$x^2 + 1^2 = a^2$$

$$\sqrt{x^2+1^2} = a$$

$$\sec y = \frac{\sqrt{x^2+1^2}}{1}$$

$$\therefore \frac{dy}{dx} = \frac{1}{1+x^2}$$

$$\frac{dy}{dx} tan^{-1} u = \frac{1}{1+u^2} \frac{du}{dx}$$

Example

Find the derivative of $y = \sin^{-1}\left(3x^2\right)$.

$$y' = \frac{1}{\sqrt{1-(3x^2)^2}} \cdot 6x$$

$$= \frac{6x}{\sqrt{1-9x^4}}$$

$$\frac{d}{dx}[\arctan(x^3)] =$$

$$\frac{1}{1+(x^3)^2} \cdot 3x^2$$

$$= \frac{3x^2}{1+x^6}$$

If $g(x) = \sin^{-1}\left(\frac{x}{2}\right)$, then $\lim\limits_{x\to\sqrt{3}} \frac{g(x)-g(\sqrt{3})}{x-\sqrt{3}} =$

Means derivative of arc sin at $x = \sqrt{3}$

$$g'(x) = \frac{1}{\sqrt{1-\left(\frac{x}{2}\right)^2}} \cdot \frac{1}{2}$$

$$= \frac{1}{2} \cdot \frac{1}{\sqrt{1-\frac{x^2}{4}}}$$

$$g'(\sqrt{3}) = \frac{1}{2} \cdot \frac{1}{\sqrt{1-\frac{(\sqrt{3})^2}{4}}} = \frac{1}{2} \cdot \frac{1}{\sqrt{\frac{1}{4}}}$$

$$= \frac{1}{2} \cdot \frac{1}{\frac{1}{2}} = 1$$

Example

Find the derivative of $y = cos^{-1}(4x^3\text{-}3)$

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1-(4x^3-3)^2}} \cdot 12x^2$$

$$= \frac{-12x^2}{\sqrt{1-(16x^6-24x^3+9)}}$$

$$= \frac{-12x^2}{\sqrt{-16x^6+24y^3-8}}$$

If $\arccos(3x) = \ln y$, then $\frac{dy}{dx} =$

Simplify first

$$e^{\cos^{-1}(3x)} = e^{\ln y}$$

$$e^{\cos^{-1}(3x)} = y$$

$$\frac{dy}{dx} = e^{\cos^{-1}(3x)} \cdot \frac{-1}{\sqrt{1-(3x)^2}} \cdot 3 = \frac{-3e^{\cos^{-1}(3x)}}{\sqrt{1-9x^2}}$$

Chart to memeorize derivatives of inverse trig functions

These are the three main inverse trig derivatives studied in this course. They can all de developed the same way as arcsin!

$$\frac{d(\sin^{-1} u)}{dx} = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$$

$$\frac{d(\cos^{-1} u)}{dx} = \frac{-1}{\sqrt{1-u^2}}\frac{du}{dx}$$

$$\frac{d(\tan^{-1} u)}{dx} = \frac{1}{1+u^2}\frac{du}{dx}$$

$$\boxed{\frac{d}{dx} arc\csc x = \frac{d}{dx}\csc^{-1} x = \frac{-1}{|x|\sqrt{x^2-1}}}$$

$$\boxed{\frac{d}{dx} arc\sec x = \frac{d}{dx}\sec^{-1} x = \frac{1}{|x|\sqrt{x^2-1}}}$$

$$\boxed{\frac{d}{dx} arc\cot x = \frac{d}{dx}\cot^{-1} x = \frac{-1}{1+x^2}}$$

Example

Find the derivative of $y = csc^{-1}(5x^2+1)$

$$y' = \frac{-1}{|5x^2+1|\sqrt{(5x^2+1)^2-1}} \cdot 10x$$

$$y' = \frac{-10x}{(5x^2+1)\sqrt{25x^4+10x^2}}$$

Example: $y = sec^{-1}(3e^{4x})$

$$y' = \frac{1}{|3e^{4x}|\sqrt{(3e^{4x})^2 - 1}} \cdot 3e^{4x} \cdot 4$$

$$y' = \frac{12\cancel{e^{4x}}}{(3\cancel{e^{4x}})\sqrt{9e^{8x} - 1}}$$

$$y' = \frac{4}{\sqrt{9e^{8x} - 1}}$$

$$\frac{d}{dt}[\cot^{-1}(\cos t)] = \frac{-1}{1+(\cos t)^2} \cdot -\sin t$$

$$= \frac{\sin t}{1+\cos^2 t}$$

1. If $f(x) = e^{2x}\tan^{-1}(x)$, then $f'(1) =$

(A) $\frac{e^2}{2}$ (B) $\frac{e^2\pi}{4}$ (C) e^2 (D) $\frac{e^2\pi}{2}$ (E) $\frac{e^2(\pi+1)}{2}$

$$f'(x) = e^{2x}\cdot\frac{1}{1+x^2} + \tan^{-1}(x)\cdot 2e^{2x}$$

$$= e^{2x}\left[\left(\frac{1}{1+x^2}\right) + 2\tan^{-1}(x)\right]$$

$$= e^2\left[\frac{1}{2} + \pi/2\right]$$

Calculator Allowed

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Question 3

A particle moves along the y-axis so that its velocity v at time $t \geq 0$ is given by $v(t) = 1 - \tan^{-1}\left(e^t\right)$. At time $t = 0$, the particle is at $y = -1$. (Note: $\tan^{-1} x = \arctan x$)

(a) Find the acceleration of the particle at time $t = 2$.

(b) Is the speed of the particle increasing or decreasing at time $t = 2$? Give a reason for your answer.

(c) Find the time $t \geq 0$ at which the particle reaches its highest point. Justify your answer.

(d) Find the position of the particle at time $t = 2$. Is the particle moving toward the origin or away from the origin at time $t = 2$? Justify your answer.

enter $v(t)$ into function #1 on calculator

a) $a(2) = v'(2) = -0.133$

b) $a(3) = -0.133$

using calculator find $v(3) = -0.436$

since $a(3) < 0$ and $v(3) < 0$ particle is speeding up at $t = 3$

e) $v(t) = 0$ when $\tan^{-1}(e^t) = 1$

$\therefore\ t = \ln(\tan 1) \approx 0.443$ only critical value

$\therefore$ max height at $t = 0.443$.

d) $Y(2) = -1 + \int_0^2 v(t)\,dt$

use calculator

$= -1 + (-0.3606887)$

$= -1.361$

since below origin at $t = 2$ and $v(2)$ is negative particle is moving away from origin.

Assignment Handout

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