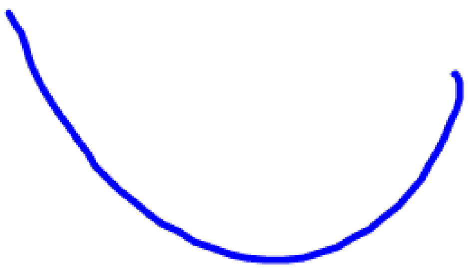
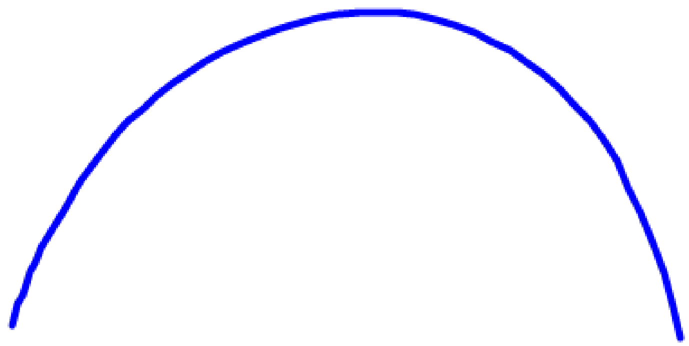
4.3 Concavity and the Second Derivative Test






recall

A function is said to be **increasing** if the graph is **"going uphill".**

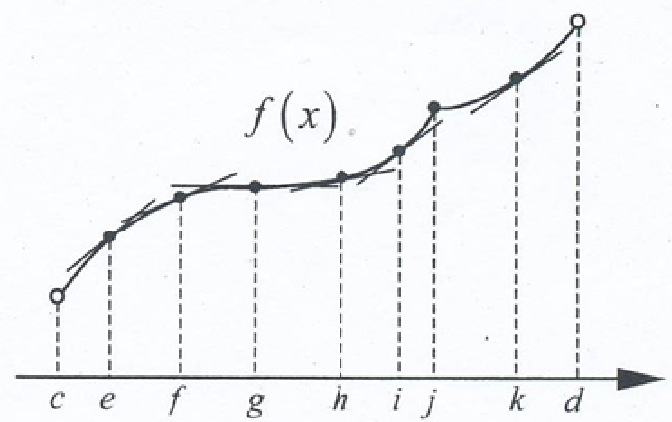


Figure 3

If $f'(x) > 0$ for all x in an open interval, then $f(x)$ is increasing on this open interval.

A function is said to be decreasing if the graph is "going downhill".

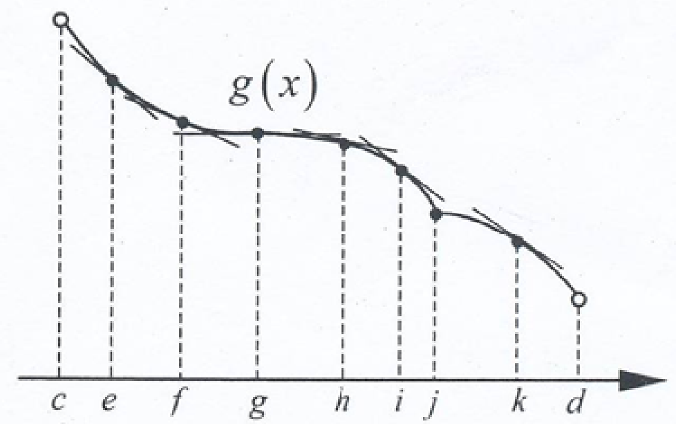


Figure 4

If $f'(x) < 0$ for all x in an open interval, then $f(x)$ is decreasing on this open interval.

A function is said to be **concave up** if the graph is shaped like a **"valley".**

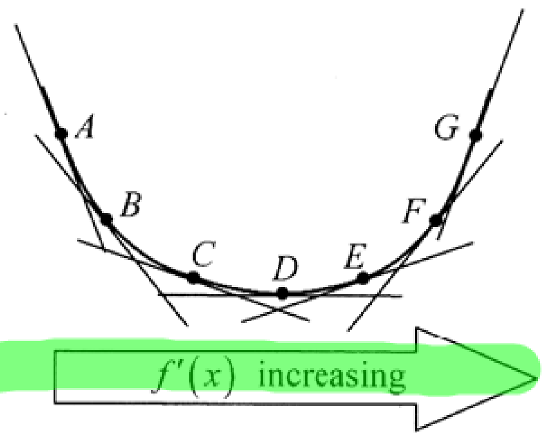


Figure 1
$f'(x)$ is increasing, thus $f''(x)$ is positive. The curve is concave up.

CU
$f'' > 0$
f' inc

If $f''(x) > 0$, for all x in an interval I, then $f(x)$ is concave up.

A function is said to be **concave down** if the graph is shaped like a **"hill".**

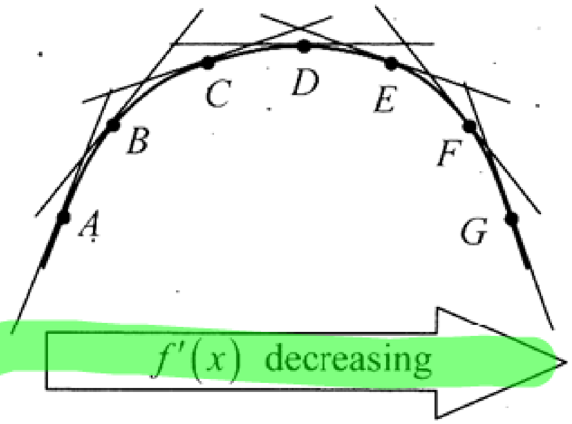


Figure 2

$f'(x)$ is decreasing, thus $f''(x)$ is negative. The curve is concave down.

CD

$f'' < 0$

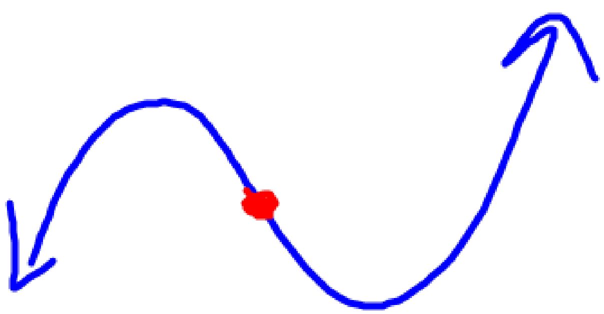
f' dec

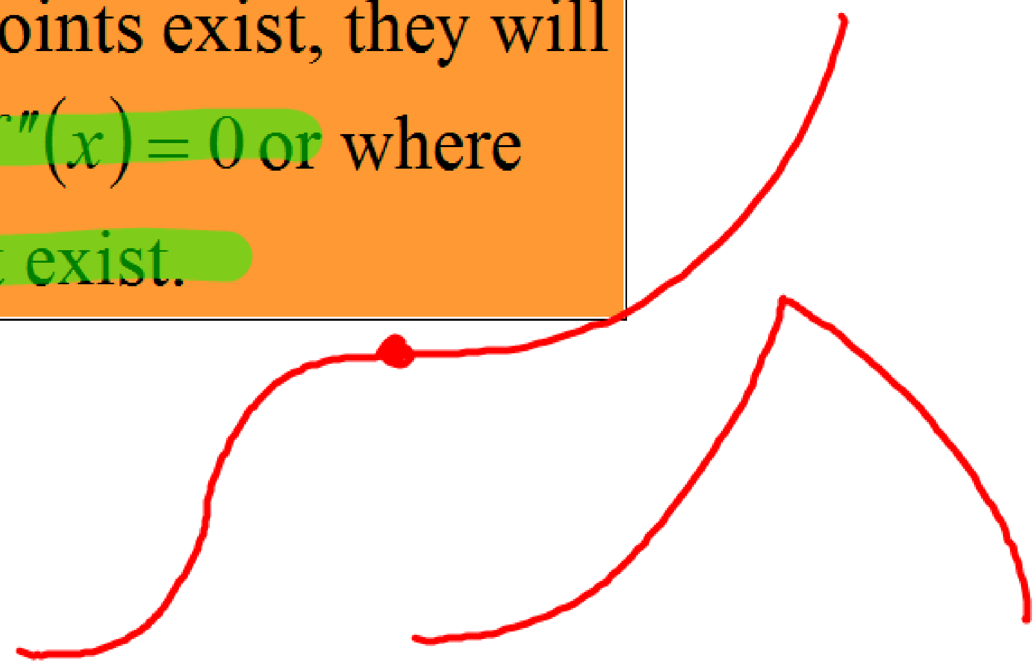
If $f''(x) < 0$, for all x in an interval I, then $f(x)$ is concave down.

Inflection Points

Inflection Points or IP's are points where the concavity of a function changes.

If inflection points exist, they will occur where $f''(x) = 0$ or where $f''(x)$ does not exist.





Ex.1 Determine where the following functions are concave up, concave down and find all IP's

a) $y = x^3 - 3x^2 + 4x - 5$

$y' = 3x^2 - 6x + 4$

$y'' = 6x - 6$

$y'' = 0$

$\boxed{6x - 6 = 0}$

$6x = 6$

$x = 1$

f CU

$(1, \infty)$

f CD

$(-\infty, 1)$

IP $f(1) = -3$

$-$ 0 $+$ y''

0 1 2

b) $f(x) = x^3 - 6x^2 - 2x + 4$

c) $y = \dfrac{x}{x^2+1}$

$$y' = \frac{(x^2+1)(1) - x(2x)}{(x^2+1)^2}$$

$$y' = \frac{1-x^2}{(x^2+1)^2}$$

$$y'' = \frac{(x^2+1)^{\cancel{2}}(-2x) - (1-x^2)\cdot 2\,\cancel{(x^2+1)}^{1}\cdot 2x}{(x^2+1)^{\cancel{4}\,3}}$$

$$= \frac{-2x^3 - 2x - 4x + 4x^3}{(x^2+1)^3}$$

$$y'' = \frac{2x^3-6x}{(x^2+1)^3}$$

$$\underline{y''=0}$$

$$2x^3-6x=0$$

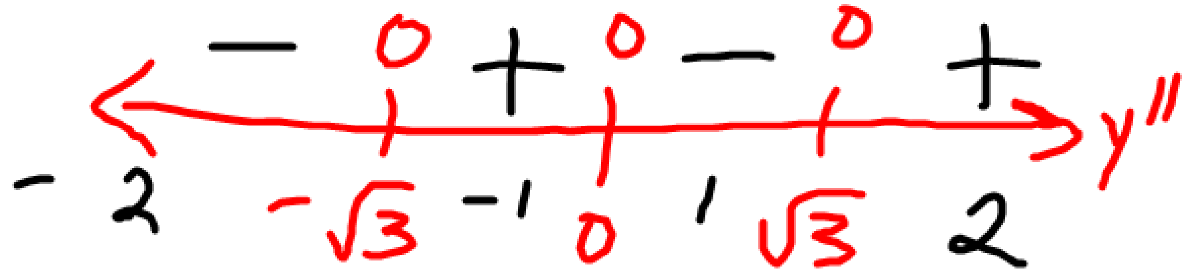
$$2x(x^2-3)=0$$

$$\boxed{2x(x-\sqrt{3})(x+\sqrt{3})}=0$$

$x=0 \quad x=-\sqrt{3}$

$x=\sqrt{3}$

$\cancel{y''\ \infty}$



$$\underline{f \text{ CU}}$$

$$(-\sqrt{3},0)\cup(\sqrt{3},\infty)$$

$$\underline{f \text{ CD}}$$

$$(-\infty,-\sqrt{3})\cup(0,\sqrt{3})$$

IP

$f(0) = 0$

$f(\sqrt{3}) = \frac{\sqrt{3}}{4}$

$f(-\sqrt{3}) = \frac{-\sqrt{3}}{4}$

$f < 0$

below x axis

$f' < 0$

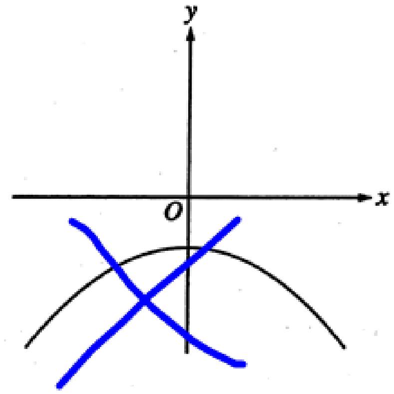
dec

$f'' < 0$

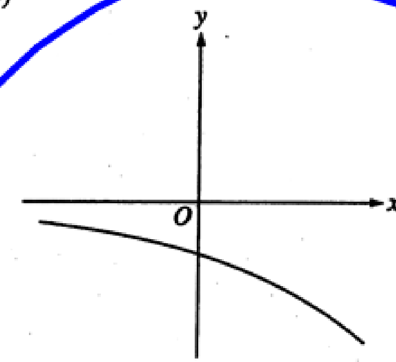
(D)

10. The function f has the property that $f(x)$, $f'(x)$, and $f''(x)$ are negative for all real values x. Which of the following could be the graph of f ?

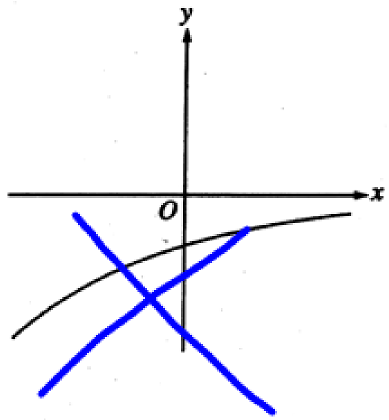
(A)



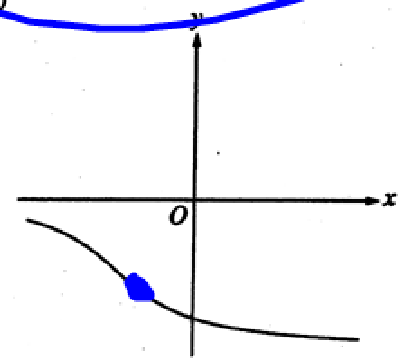
(B)



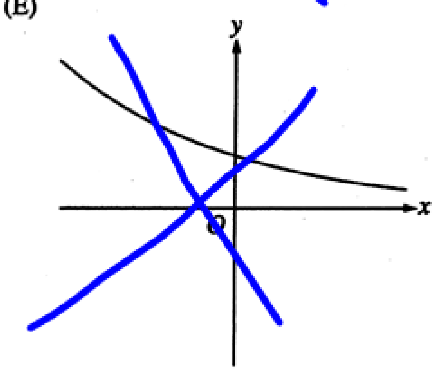
(C)

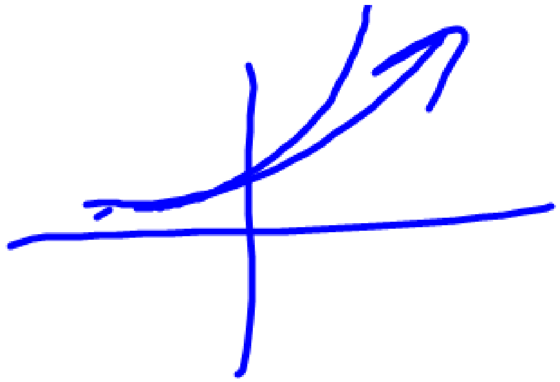


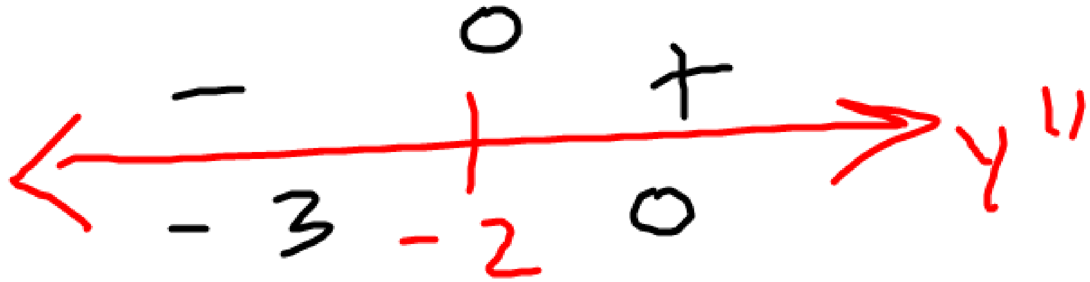
(D)



(E)







2003 AP MC Question No Calculator

17 Let f be the function given by $f(x) = 2xe^x$. The graph of f is concave down when

(A) $x < -2$ (B) $x > -2$ (C) $x < -1$ (D) $x > -1$ (E) $x < 0$

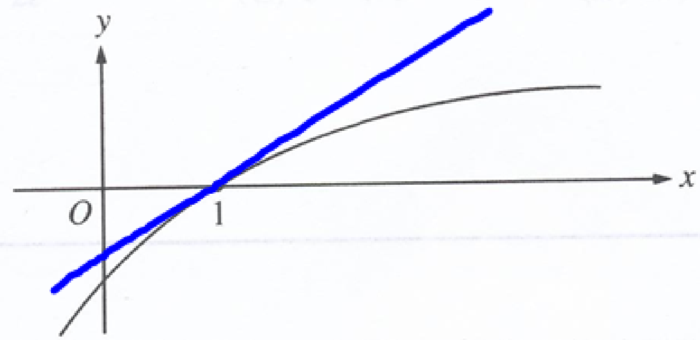
$f'(x) = 2xe^x + e^x(2)$

$f' = 2e^x(x+1)$

$f'' = 2e^x(1) + (x+1)(2e^x)$

$= \boxed{2e^x(x+2)}$

$x = -2 \quad f'' = 0$



17. The graph of a twice-differentiable function f is shown in the figure above. Which of the following is true?

(A) $f(1) < f'(1) < f''(1)$
(B) $f(1) < f''(1) < f'(1)$
(C) $f'(1) < f(1) < f''(1)$
(D) $f''(1) < f(1) < f'(1)$
(E) $f''(1) < f'(1) < f(1)$

$f(1) = 0$

$f'(1) > 0$

$f''(1) < 0$

$f(x) = kx^{1/2} - \ln x$

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Question 6

Let f be the function defined by $f(x) = k\sqrt{x} - \ln x$ for $x > 0$, where k is a positive constant.

(a) Find $f'(x)$ and $f''(x)$.

(b) For what value of the constant k does f have a critical point at $x = 1$? For this value of k, determine whether f has a relative minimum, relative maximum, or neither at $x = 1$. Justify your answer.

(c) For a certain value of the constant k, the graph of f has a point of inflection on the x-axis. Find this value of k.

a) $f' = \frac{k}{2}x^{-1/2} - \frac{1}{x}$

$f'' = \frac{-k}{4}x^{-3/2} + x^{-2}$

b) $f' = \dfrac{k}{2\sqrt{x}} - \dfrac{1}{x}$

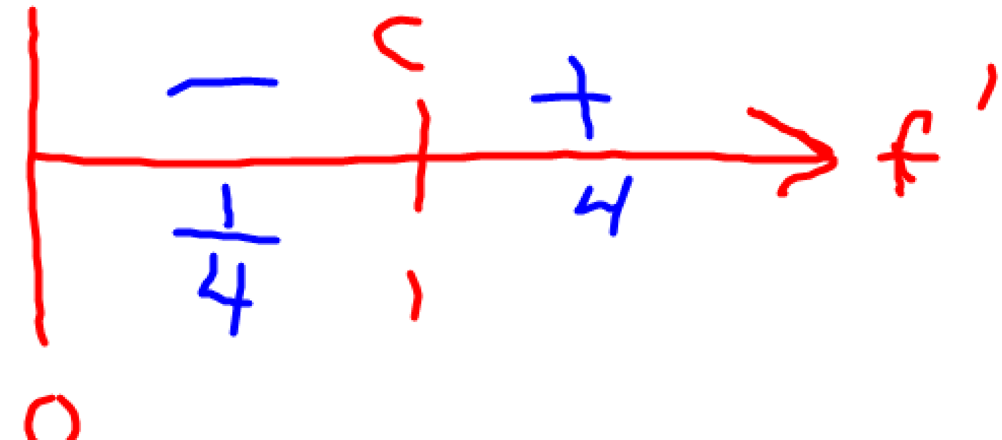
$0 = \dfrac{k}{2\sqrt{1}} - \dfrac{1}{1}$

$1 = \dfrac{k}{2}$

$2 = k$

$2 \quad - \quad 4$

$f' = \dfrac{1}{\sqrt{x}} - \dfrac{1}{x}$



$x = 1$ Rel Min

b/c f' Δ's − to + at $x = 1$.

IP

$$4'' = \frac{-k}{4x^{3/2}} + \frac{1}{x^2}$$

$$\frac{k}{4x^{3/2}} = \frac{1}{x^2}$$

$$k = \frac{4x^{3/2}}{x^2}$$

$$k = \frac{4}{\sqrt{x}}$$

xint

$$0 = k\sqrt{x} - \ln x$$

$$\ln x = k\sqrt{x}$$

$$\frac{\ln x}{\sqrt{x}} = k$$

$$\frac{4}{\sqrt{x}} = \frac{\ln x}{\sqrt{x}}$$

$$4\sqrt{x} = \sqrt{x}\ln x$$

$$4\sqrt{x} - \sqrt{x}\ln x = 0$$

$$\sqrt{x}(4 - \ln x) = 0$$

$\sqrt{x} = 0$ OR $4 - \ln x = 0$

$\cancel{x = 0}$

$$4 = \ln x$$

$$4 = \log_e x$$

$$e^4 = x$$

$$4 = \ln x$$

$$e^4 = e^{\ln x}$$

$$\log_2 2^3$$

$$K = \frac{4}{\sqrt{x}}$$

$$= \frac{4}{\sqrt{e^4}}$$

$$= \frac{4}{e^2}$$

Second Derivative Test For Local Maximums and Local Minimums

We can use the **second derivative** to also help us find **local maximums** and **minimums.**

Suppose *f(x)* is a continuous function in an interval containing "c" and "c" a critical number:

$$f'(c) = 0 \text{ and } f''(c) < 0, \text{ then "c" is a local maximum.}$$

$$f'(c) = 0 \text{ and } f''(c) > 0, \text{ then "c" is a local minimum.}$$

Example 1: Find any local extrema of $f(x) = x^4 - 8x^2$ using the Second Derivative Test.

$f' = 4x^3 - 16x$

$4x^3 - 16x = 0$

$4x(x^2 - 4) = 0$

$4x(x-2)(x+2) = 0$

$x = 0$

$x = 2$

$x = -2$

$f'' = 12x^2 - 16$

$f''(0) = < 0$

$x = 0$ max

$f''(2) = > 0$

$x = 2$ min

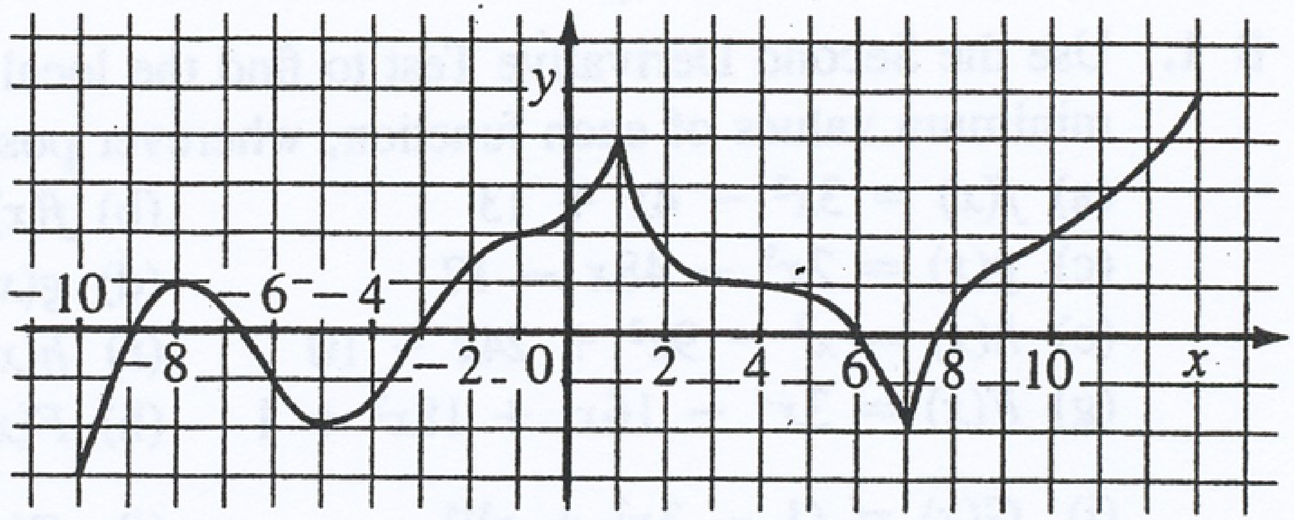
$f''(-2) = > 0$

$x = -2$ min

Assignment
Handout
1,2c,d,e,4
Page 246 Calc 30 Text
#'s 6,7,10,12,15-19

A **1.** (a) State the intervals on which f is concave upward or concave downward.

(b) State the coordinates of the points of inflection.



2nd Derivative Test For Max/Mins.

Some values of a twice differentiable function, $f(x)$, and its first and second derivatives, $f'(x)$ and $f''(x)$ respectively, are given in the table below. For example, $f'(3) = -2$. Use the table to answer the questions that follow.

x	1	2	3	4
$f(x)$	4	5	1	2
$f'(x)$	0	0	–2	0
$f''(x)$	2	0	0	–3

1. Does $A(x) = (f(x))^2$ have a critical point at $x = 4$? If $A(x)$ does have a critical point, can you determine whether it is a local maximum, local minimum, or neither? Explain your answer.

2. Does $B(x) = f(x^2)$ have a critical point at $x = 2$? If $B(x)$ does have a critical point, can you determine whether it is a local maximum, local minimum, or neither? Explain your answer.

3. Does $C(x) = f(f(x))$ have a critical point at $x = 3$? If $C(x)$ does have a critical point, can you determine whether it is a local maximum, local minimum, or neither? Explain your answer.

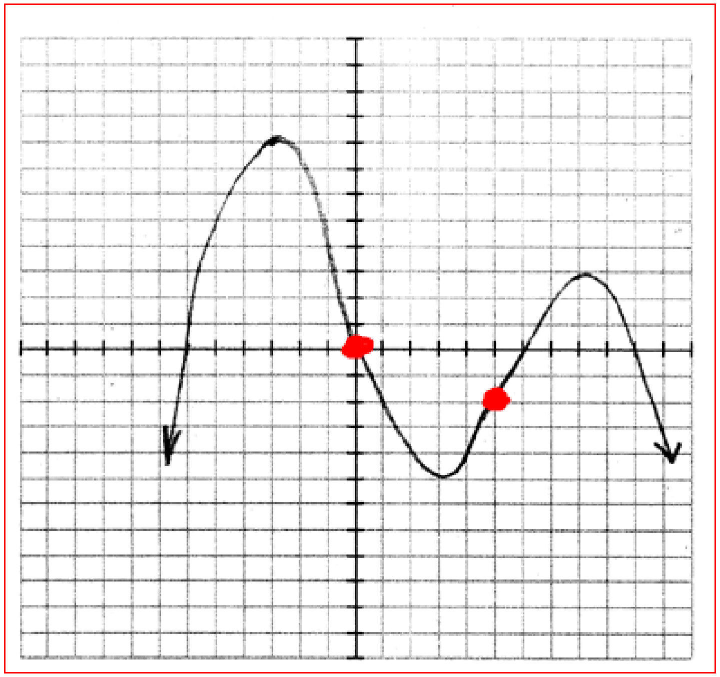
4. Does $D(x) = f(4x - 2)$ have a critical point at $x = 1$? If $D(x)$ does have a critical point, can you determine whether it is a local maximum, local minimum, or neither? Explain your answer.

5. Does $E(x) = f(x + 3)$ have a critical point at $x = 0$? If $E(x)$ does have a critical point, can you determine whether it is a local maximum, local minimum, or neither? Explain your answer.

4.3 Connecting Graphs of $f(x), f'(x), f''(x)$

Day 2

y=f(x)



1. Where is the function increasing?

$(-\infty,-3) \cup (3,8)$ $f' > 0$

2. Where is the function decreasing?

$(-3,3) \cup (8,\infty)$ $f' < 0$

3. Where is the function concave up?

$(0,5)$ $f'' > 0$

4. Where is the function concave down?

$(-\infty,0) \cup (5,\infty)$ $f'' < 0$

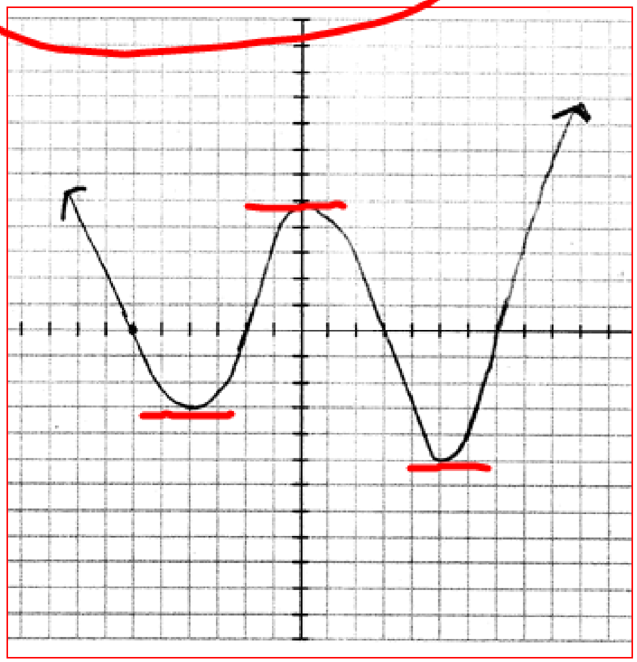
5. At what x coordinate(s) does the function have a relative max?

$x=-3$ $x=8$

6. At what x coordinate(s) does the function have a relative min?

$x=3$

y=f'(x)



Justify

1. Where is $f(x)$ increasing?

$(-\infty,-6)\cup(-2,3)\cup(7,\infty)$ $f'>0$

2. Where is $f(x)$ decreasing?

$(-6,-2)\cup(3,7)$ $f'<0$

3. At what x coordinate does $f(x)$ have a relative max?

$x=-6$ $x=3$

f' Δ's $+ \to -$

4. At what x coordinate does $f(x)$ have a relative min?

$x=-2$, $x=7$

f' Δ's $-$ to $+$.

5. Where is $f(x)$ concave up?

$(-4,0)\cup(5,\infty)$

f' inc

6. Where is $f(x)$ concave down?

$(-\infty,-4)\cup(0,5)$

f' dec

7. At what x coordinate(s) does $f(x)$ have inflection points?

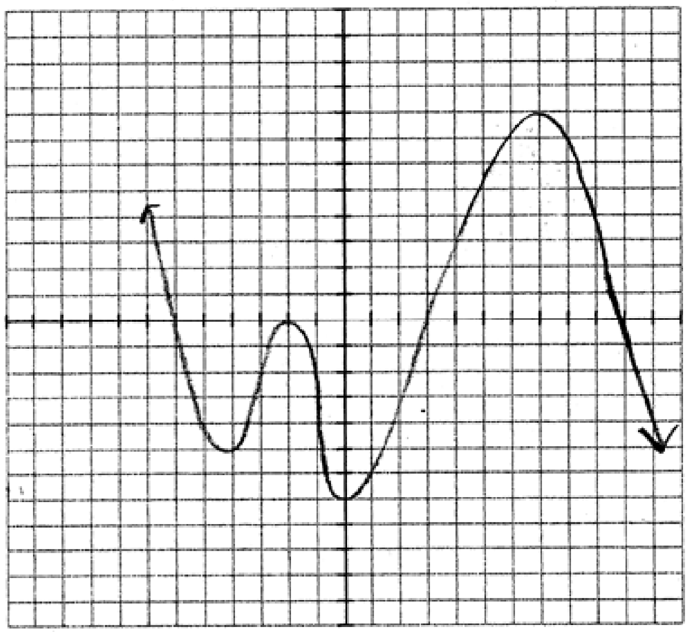
$x=-4, 5$

f' Δ's dec to inc

$x=0$

f' Δ's inc to dec

y=f(x)



1. Where is the function increasing?

2. Where is the function decreasing?

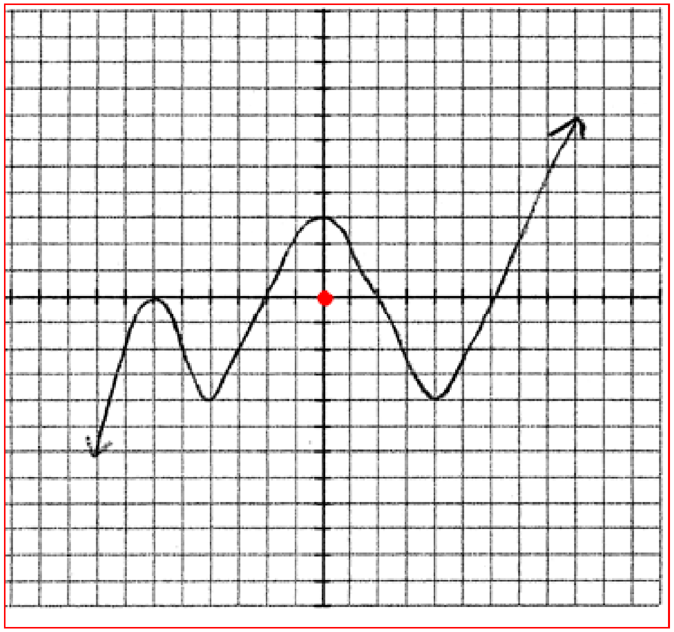
3. Where is the function concave up?

4. Where is the function concave down?

5. At what x coordinate(s) does the function have a relative max?

6. At what x coordinate(s) does the function have a relative min?

y=f'(x)



1. Where is $f(x)$ increasing?

2. Where is $f(x)$ decreasing?

3. At what x coordinate does $f(x)$ have a relative max?

4. At what x coordinate does $f(x)$ have a relative min?

5. Where is $f(x)$ concave up?

6. Where is $f(x)$ concave down?

7. At what x coordinate(s) does $f(x)$ have inflection points?

2. Shown at right is the graph of $f''(x)$ on the interval $[-2,3]$. Answer the following questions about $f(x)$. Explain your reasoning.

(a) State the open interval(s) on which $f(x)$ is concave up. $(-2,-1)\cup(1,3)$ $f''>0$

(b) State the open interval(s) on which $f(x)$ is concave down. $(-1,1)$ $f''<0$

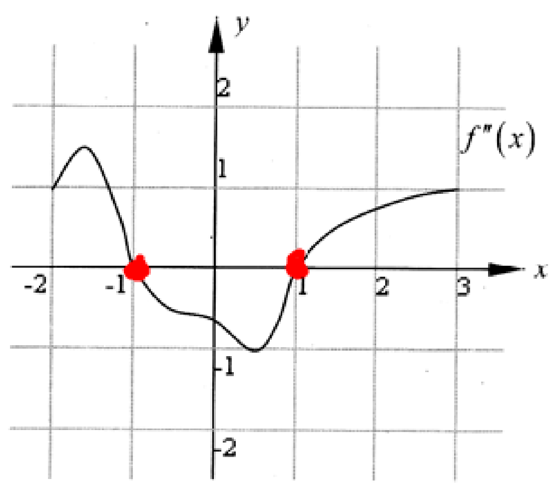
(c) State the x-value(s), if any, at which $f(x)$ has an inflection point.

$x=-1$

f'' △'s $+ \to -$

$x=1$

f'' △'s $- \to +$



Question 2

3. Shown below right is the graph of $f'(x)$ on the interval $[-3,3]$. Answer the following questions about $f(x)$. Explain your reasoning.

(a) State the open interval(s) on which $f(x)$ is increasing.

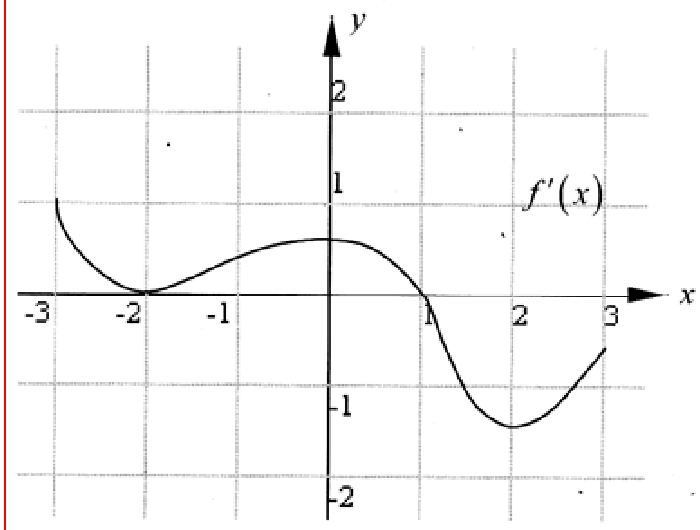
(b) State the open interval(s) on which $f(x)$ is decreasing.

(c) State the x-value(s), if any, at which $f(x)$ has a relative maximum or minimum point.

(d) State the open interval(s) on which $f(x)$ is concave up.

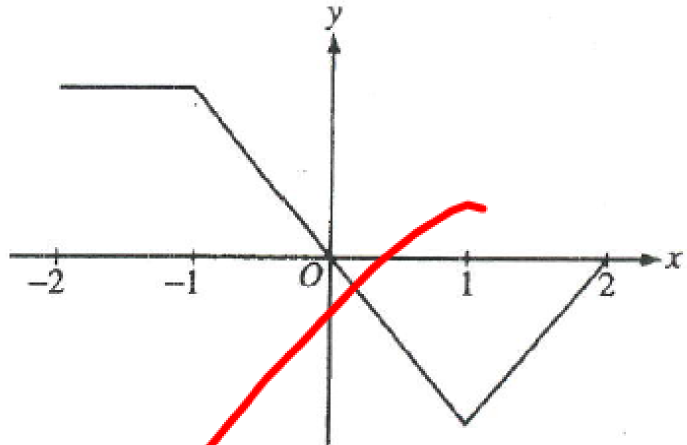
(e) State the open interval(s) on which $f(x)$ is concave down.

(f) State the x-values at which $f(x)$ has a point of inflection.



Question 3

2003 AP MC Question No Calculator



Graph of f'

7. The graph of f', the derivative of the function f, is shown above. Which of the following statements is true about f ?

(A) f is decreasing for $-1 \le x \le 1$.

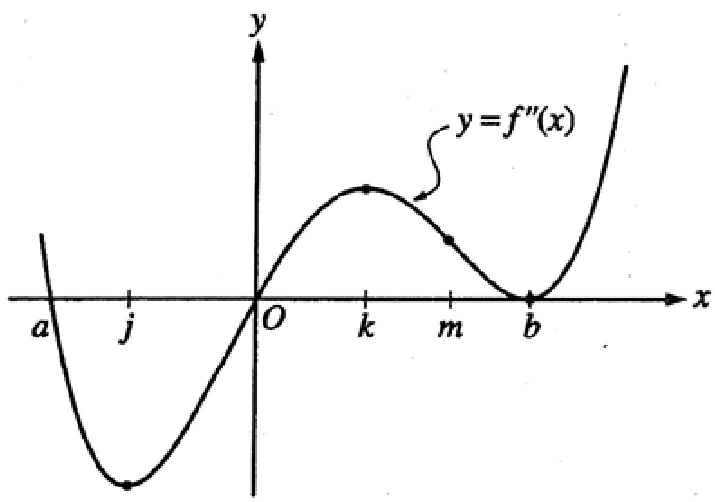
(B) f is increasing for $-2 \le x \le 0$.

(C) f is increasing for $1 \le x \le 2$.

(D) f has a local minimum at $x = 0$.

(E) f is not differentiable at $x = -1$ and $x = 1$.

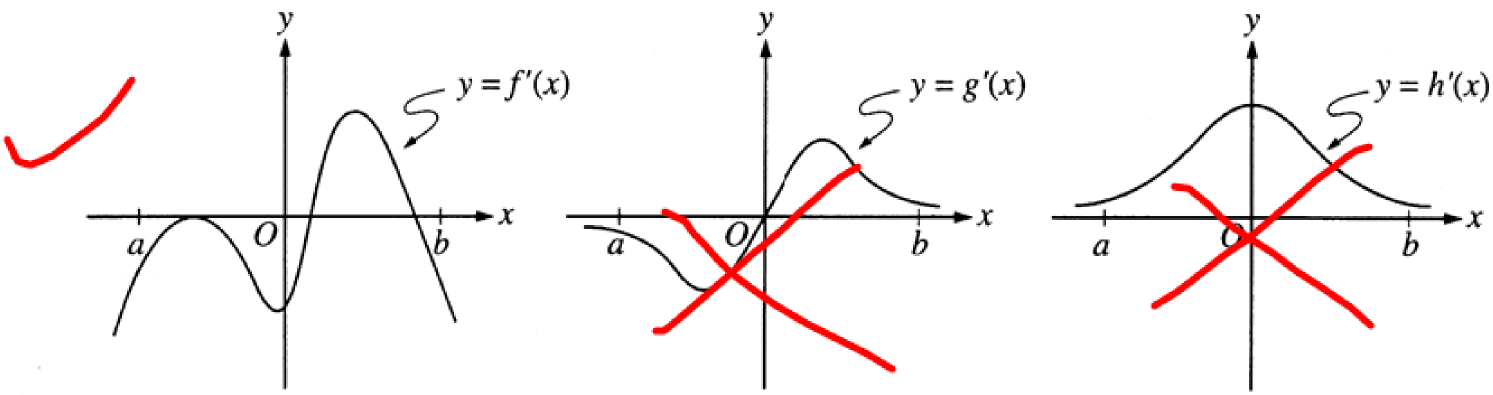
2003 AP MC Question No Calculator



21. The second derivative of the function f is given by $f''(x) = x(x-a)(x-b)^2$. The graph of f'' is shown above. For what values of x does the graph of f have a point of inflection?

(A) 0 and a only (B) 0 and m only (C) b and j only (D) 0, a, and b (E) b, j, and k

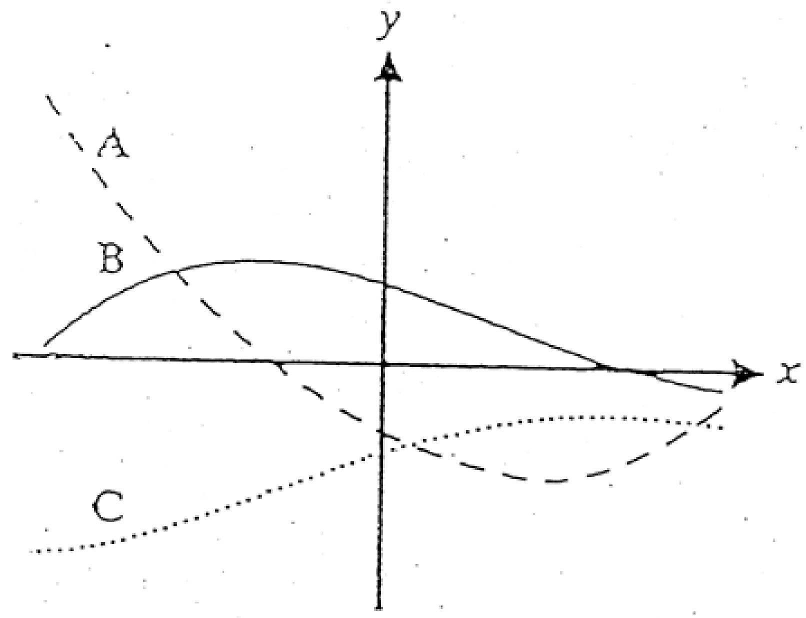
1998 AP MC Question Calculator



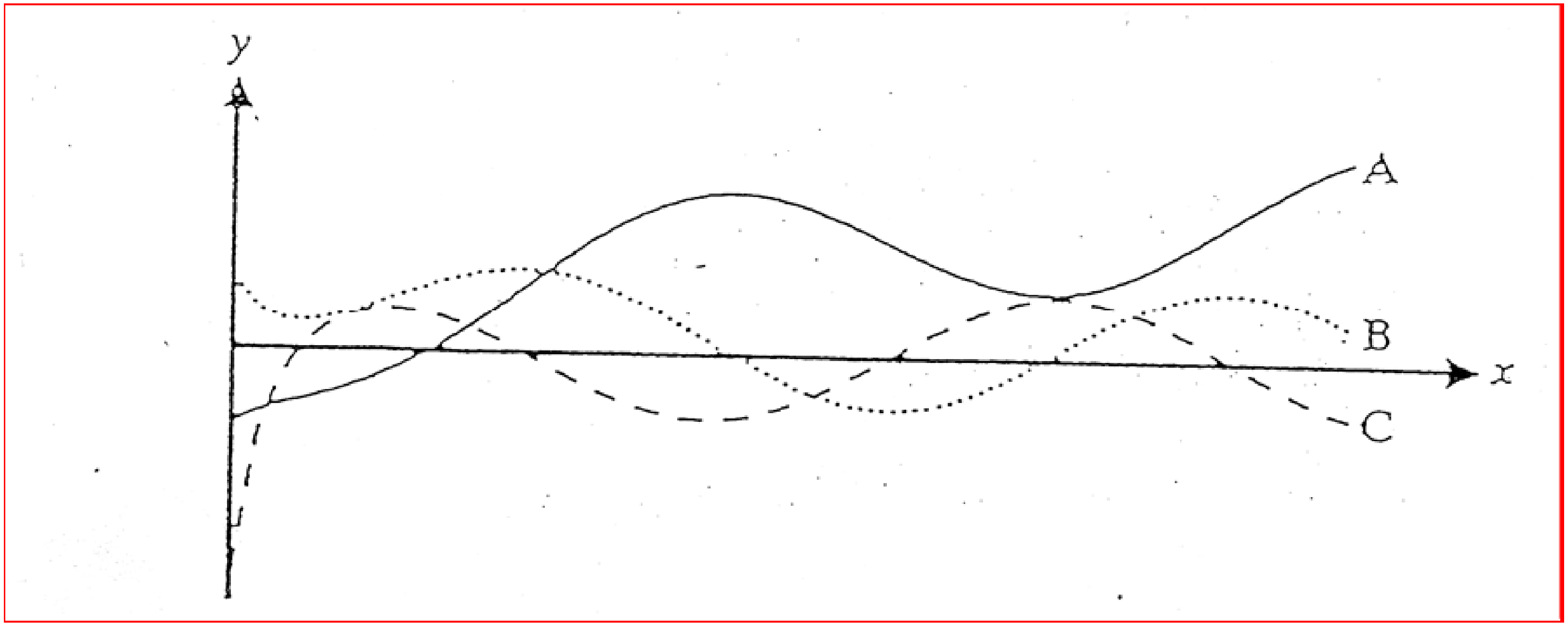
79. The graphs of the derivatives of the functions f, g, and h are shown above. Which of the functions f, g, or h have a relative maximum on the open interval $a < x < b$?

(A) f only
(B) g only
(C) h only
(D) f and g only
(E) f, g, and h

Graphs of f, f', and f'' appear below. Which is which? How can you tell'



Which is the function, first derivative and second derivative?

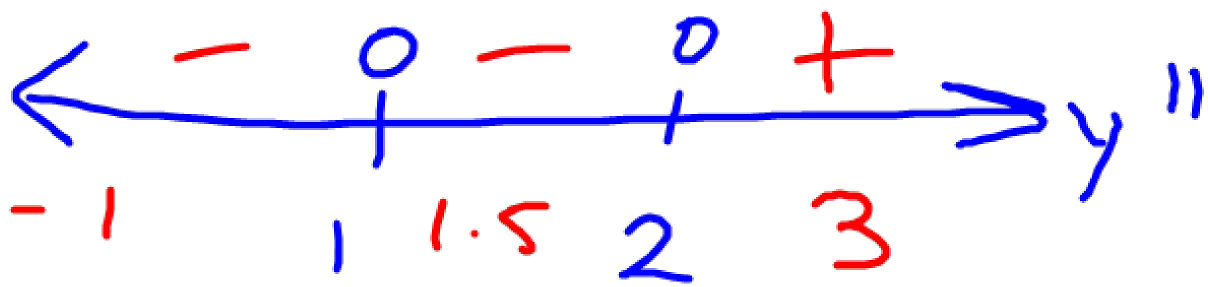


Assignment Handout
2019 AP Resource

Question 1

Suppose that *f* is differentiable and that $f'(x) = (x-1)^2(x-2)^3$. Which *x*-values have $f''(x) = 0$? What are the first coordinates of any inflection points of *f*?

$x = 1$ $\qquad$ $x = 2$



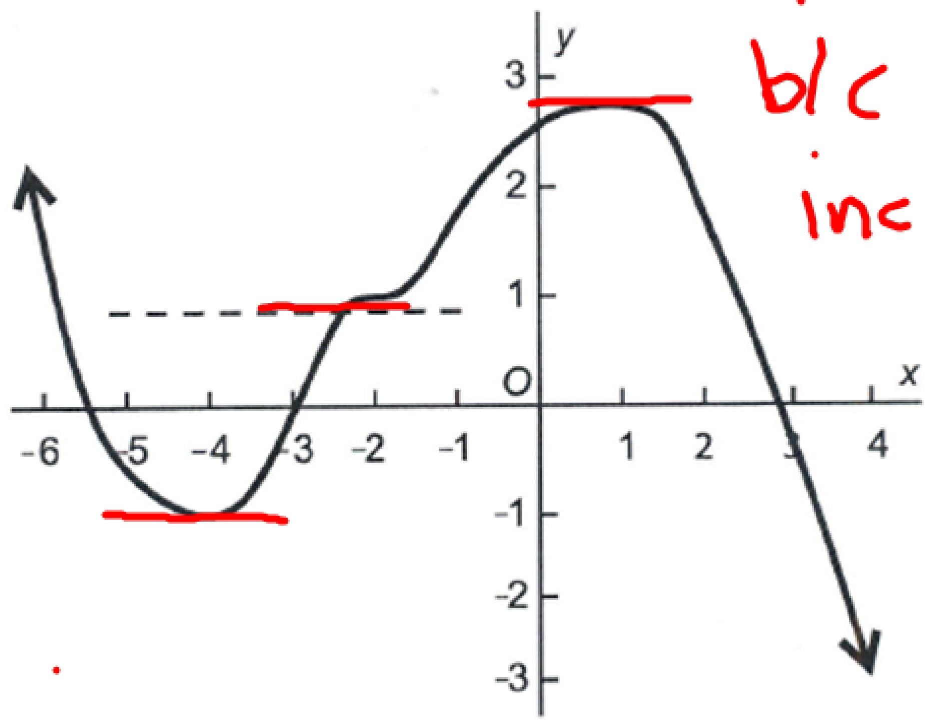
1 IP at $x = 2$.

Question 2

Suppose that g is differentiable. A graph of the *derivative* of g, that is, $y = g'(x)$, is displayed below. Use that graph to answer these questions: which x-values have $g''(x) = 0$, and what are the first coordinates of any inflection points of $g(x)$?

$x = 1$ IP
b/c g' Δ's
inc to dec.

$g''(x) = 0$
at
$x = -4, -2, 1$



$x = -4$
IP b/c
g' Δ's dec to inc

Graph of $y = g'(x)$, the derivative of $g(x)$)
(This graph has a horizontal tangent at $x = -2$.)

Finding Inflection Points

Answer the questions in the scenarios below, then compare your answer with those of your group members and discuss any differences.

Scenario 1: Suppose that $f(x) = x^4 + x^3 - 3x^2$.

a. Find the first and second derivatives of $f(x)$.

$$f' = 4x^3 + 3x^2 - 6x$$

$$f'' = 12x^2 + 6x - 6$$

b. Does $f(x)$ have any inflection points? If it does, find their coordinates and explain why they are inflection points.

$$12x^2 + 6x - 6 = 0$$

$$6(2x^2 + x - 1) = 0$$

$$6(2x - 1)(x + 1) = 0$$

$x = \frac{1}{2}$ or $x = -1$

$+ \quad 0 \quad - \quad 0 \quad +$ $\quad y''$

$-2 \quad -1 \quad 0 \quad \frac{1}{2} \quad 1$

$\underline{IP}$ $f(-1) = -3$

$\underline{IP}$ $f(\frac{1}{2}) = \frac{-9}{16}$

Scenario 2: Suppose that $g(x)=\dfrac{x^3+2}{x^2+x+1}$. Then the first derivative, $g'(x)$, is $\dfrac{x^4+2x^3+3x^2-4x-2}{(x^2+x+1)^2}$, and the second derivative, $g''(x)$, is $\dfrac{18x(x+1)}{(x^2+x+1)^3}$. Does $g(x)$ have any inflection points? If it does, find their coordinates and explain why they are inflection points.

$\underline{g''=0}$ $\cancel{\underline{g''=\infty}}$

$18x(x+1)=0$

$x=0$ OR $x=-1$

$+ \quad 0 \quad - \quad 0 \quad + \quad y''$

$-2 \quad -1 \quad -1/2 \quad 0 \quad 1$

IP
$x=-1$
$(-1, 1)$

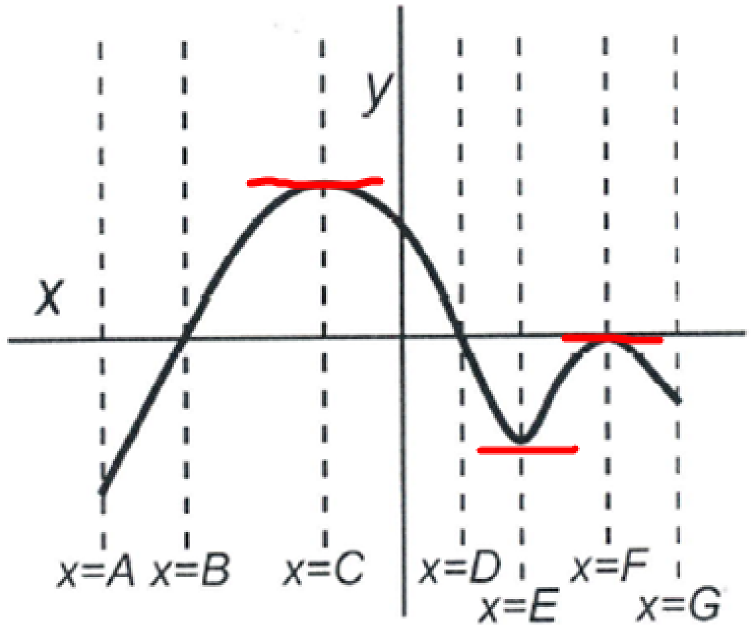
$\underline{\text{IP}}$
$x=0$ $(0, 2)$

Scenario 3: This is a graph of the *derivative* of $h(x)$, which is a function defined and continuously differentiable on the interval $[A,G]$. Use this graph of $y = h'(x)$ to answer the following questions.

$h(x)$ CU

$(A,C) \cup (E,F)$

h' inc



$h(x)$ CD

$(C,E) \cup (F,G)$

h' dec.

The graph of $y = h'(x)$

a. What are the x-coordinates of the inflection points of $h(x)$?

$x = C, F$ h' Δ's inc to dec

$x = E$ h' Δ's dec to inc

b. Justify why those x-values are inflection points.

Concavity Day 3

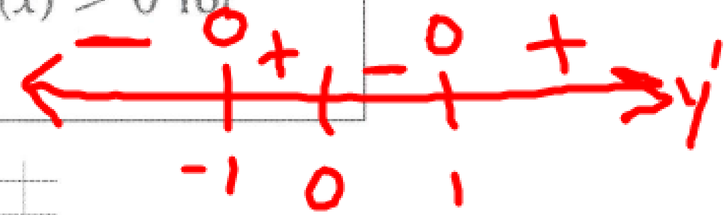
Curve Sketching

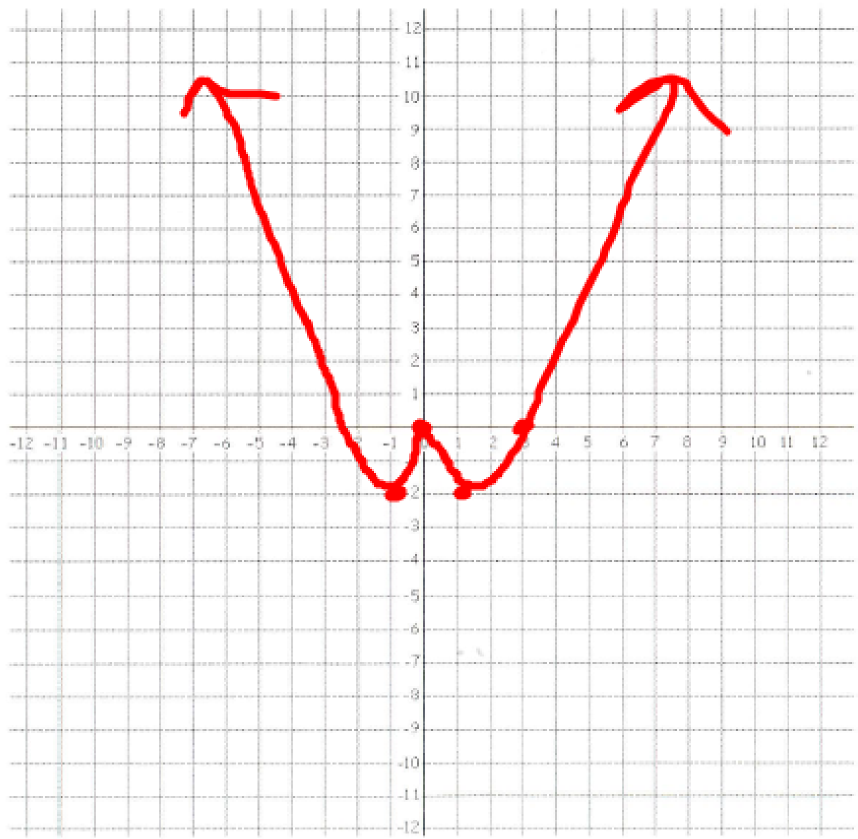
Sketch the graph of a continuous function that satisfies all of the following conditions.

(a) $f(0) = f(3) = 0, f(-1) = f(1) = -2$

(b) $f'(-1) = f'(1) = 0$ critical pts

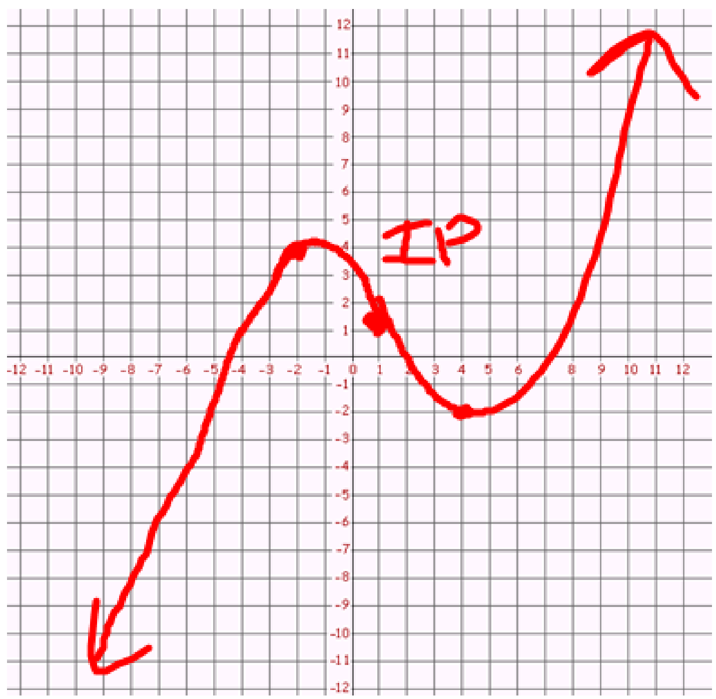
(c) $f'(x) < 0$ for $x < -1$ and for $0 < x < 1$, $f'(x) > 0$ for $-1 < x < 0$ and for $x > 1$

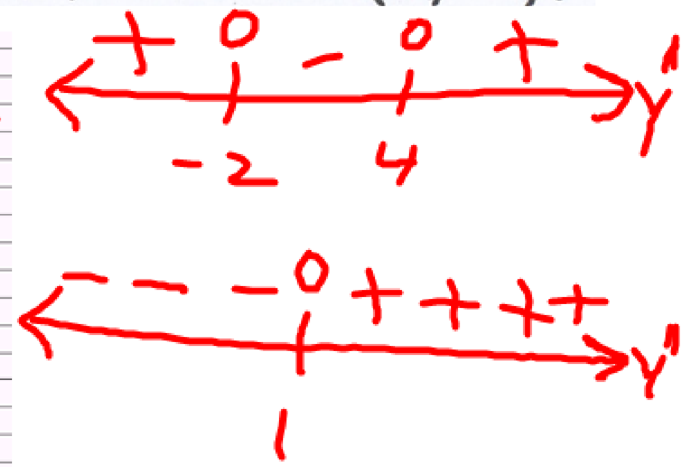




Curve Sketching

43. (i) f is continuous everywhere.
(ii) $f(-2) = 4$ and $f(4) = -2$.
(iii) $f'(-2) = f'(4) = 0$.
(iv) $f''(x) < 0$ on $(-\infty, 1)$ and $f''(x) > 0$ on $(1, \infty)$.



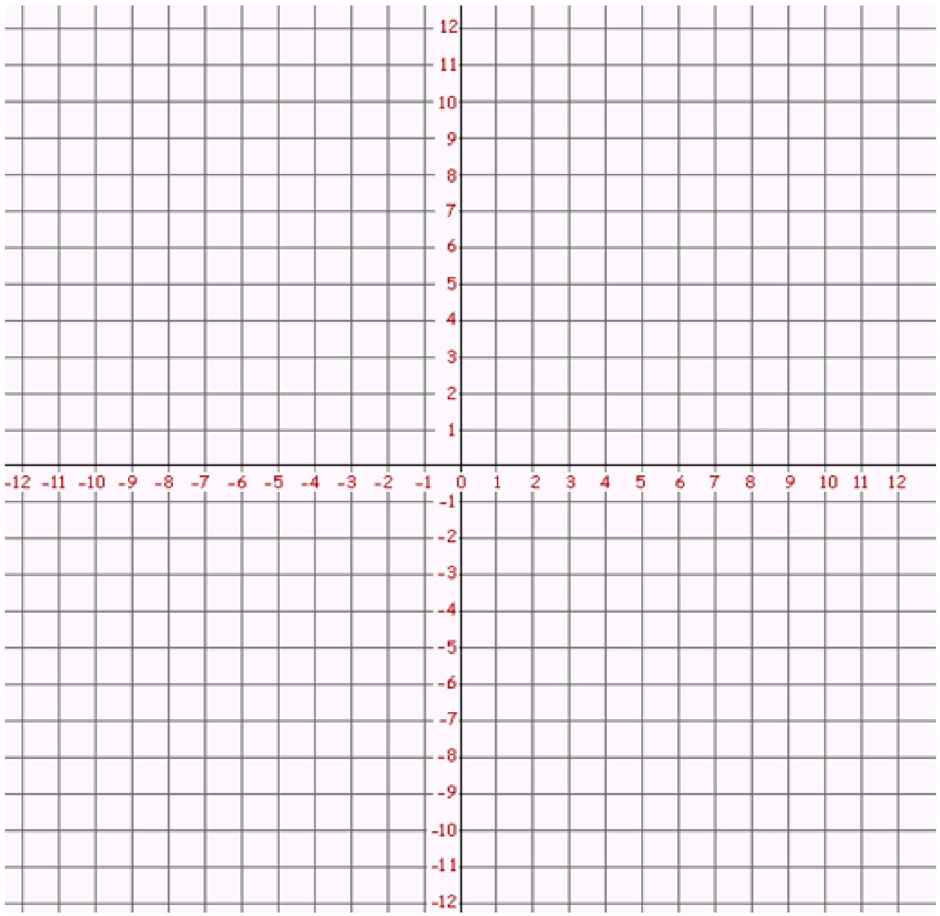


44. (i) f is continuous everywhere.

(ii) $f(0) = 2$ and $f(6) = 7$.

(iii) $f'(0) = f'(6) = 0$.

(iv) $f''(x) > 0$ on $(-\infty, 3)$ and $f''(x) < 0$ on $(3, \infty)$.



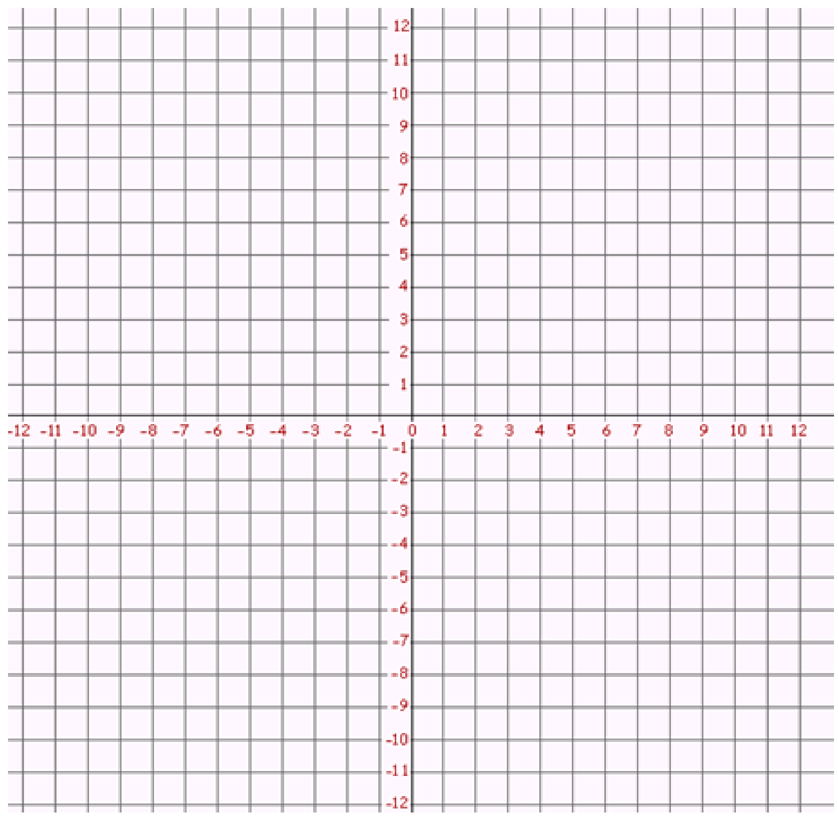
45. (i) f is continuous everywhere.

(ii) $f(-1) = 2$, $f(3) = 5$, and $f(7) = 1$.

(iii) $f'(-1) = f'(3) = f'(7) = 0$.

(iv) $f'(x) > 0$ on $(-1, 3) \cup (7, \infty)$.

(v) $f''(x) > 0$ on $(-\infty, 1) \cup (5, \infty)$ and $f''(x) < 0$ on $(1, 5)$.



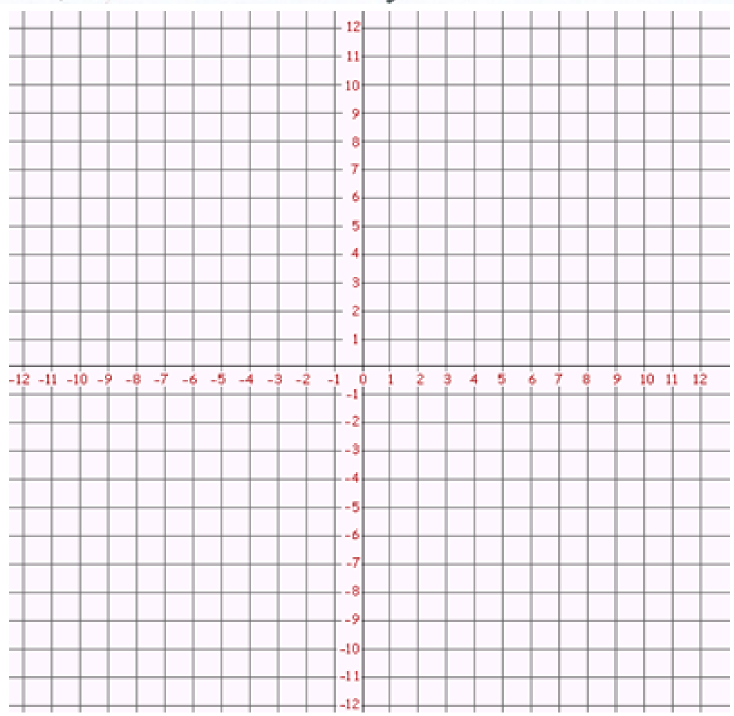
46. (i) f is continuous everywhere.

(ii) $f(-5) = 5$, $f(0) = 3$, and $f(5) = 10$.

(iii) $f'(-5) = f'(0) = f'(5) = 0$.

(iv) $f'(x) < 0$ on $(-5, 0) \cup (5, \infty)$.

(v) $f''(x) > 0$ on $(-2, 3)$ and $f''(x) < 0$ on $(-\infty, -2) \cup (3, \infty)$.



*

Ex. For the following function find:

a) intervals and of increase and decrease

b) local maxs and mins

c) intervals of concavity

d) Inflection points

e) sketch

$$f(x) = 2x^3 - 9x^2 - 108x + 2$$

a)

$f' = 6x^2 - 18x - 108$

$6x^2 - 18x - 108 = 0$

$6(x^2 - 3x - 18) = 0$

$\boxed{6(x-6)(x+3) = 0}$

$x = 6 \quad x = -3$

y': $+ \quad 0 \quad - \quad 0 \quad +$

$-4 \quad -3 \quad 0 \quad 6 \quad 7$

b)

INC	Dec
$(-\infty, -3) \cup (6, \infty)$	$(-3, 6)$

Rel Max

$f(-3) = 191$

Rel min

$f(6) = -538$

c) $\boxed{f'' = 12x - 18}$

$12x - 18 = 0$

$x = \frac{3}{2}$

$$\begin{array}{ccccc} & - & 0 & + & \\ \longleftarrow & & | & & \longrightarrow \; y'' \\ & 0 & \frac{3}{2} & 2 & \end{array}$$

$\underline{CD}$

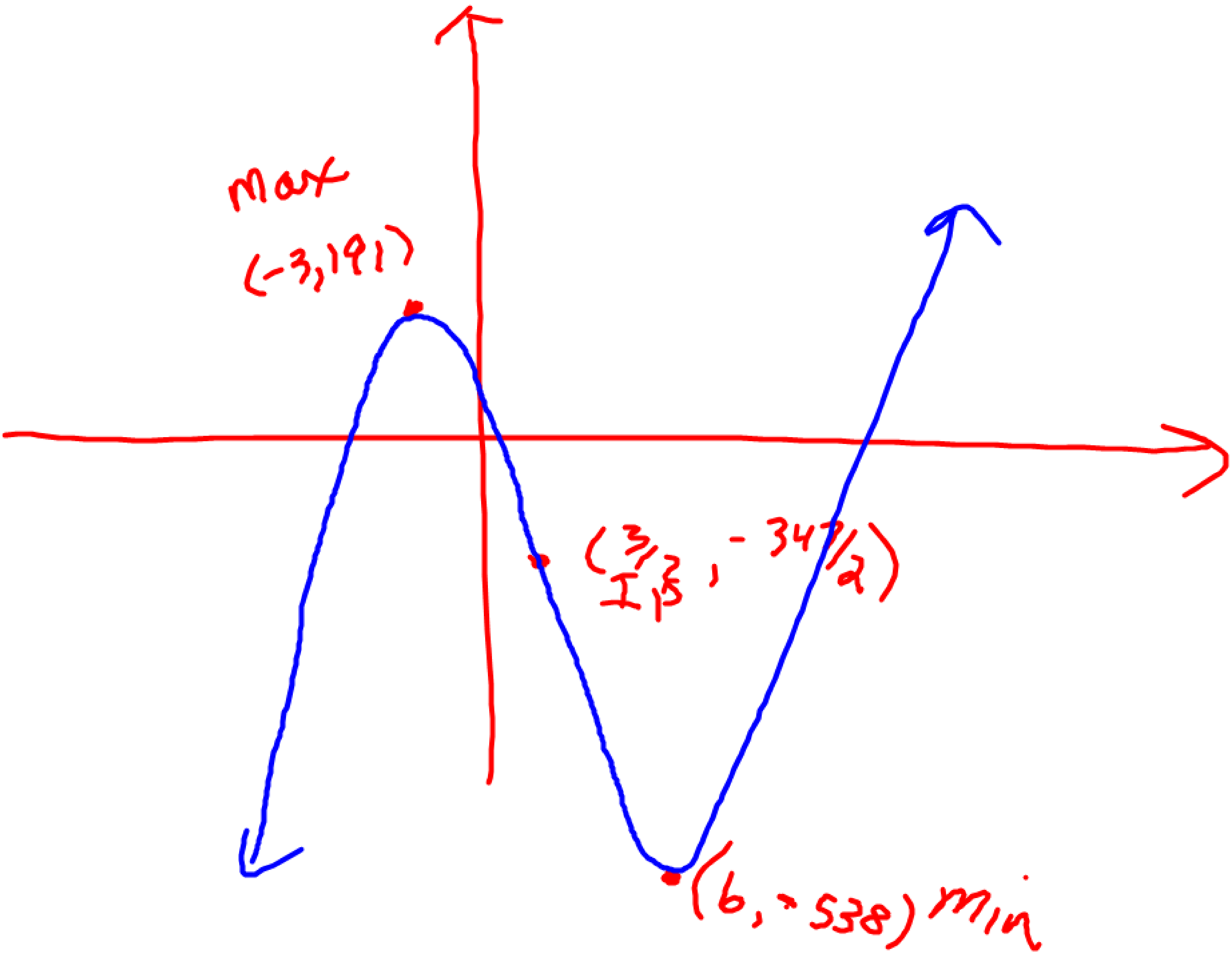
$(-\infty, \frac{3}{2})$

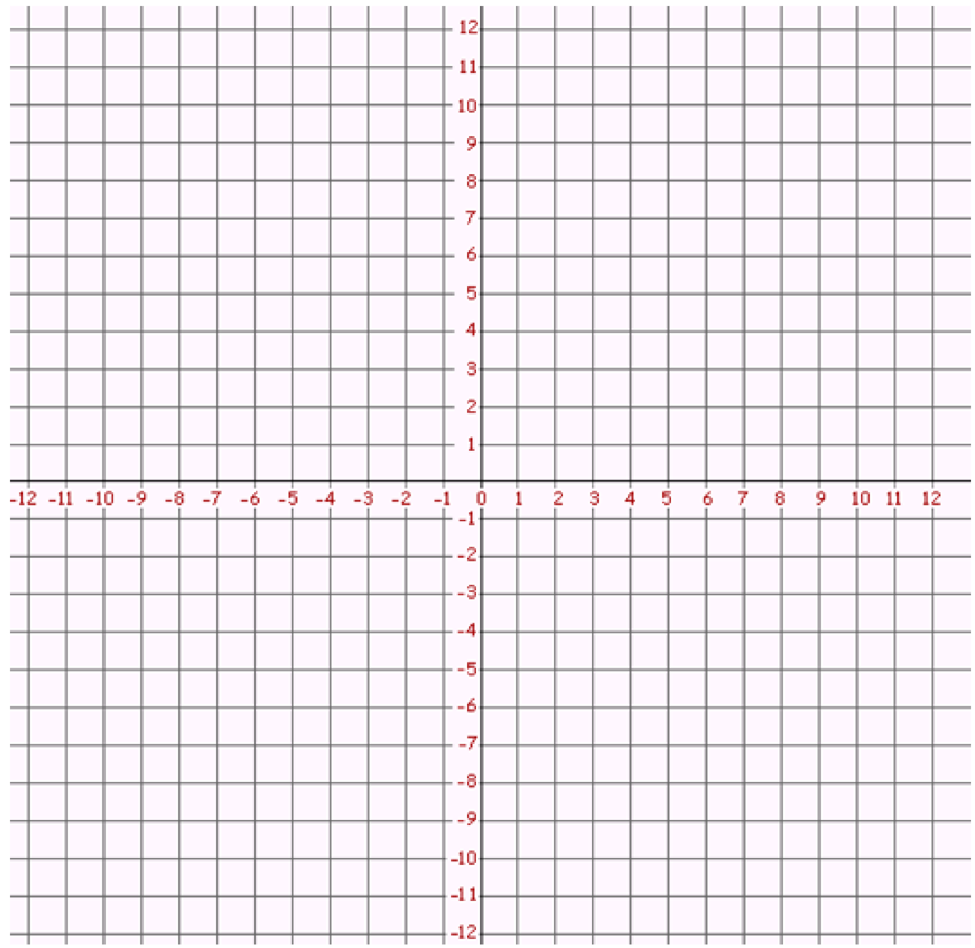
$\underline{CU}$

$(\frac{3}{2}, \infty)$

$\underline{IP}$

$f(\frac{3}{2}) = \frac{-347}{2} = -173.5$


max
(-3, 191)
(3/2, -347/2)
I.P.
(6, -538) min

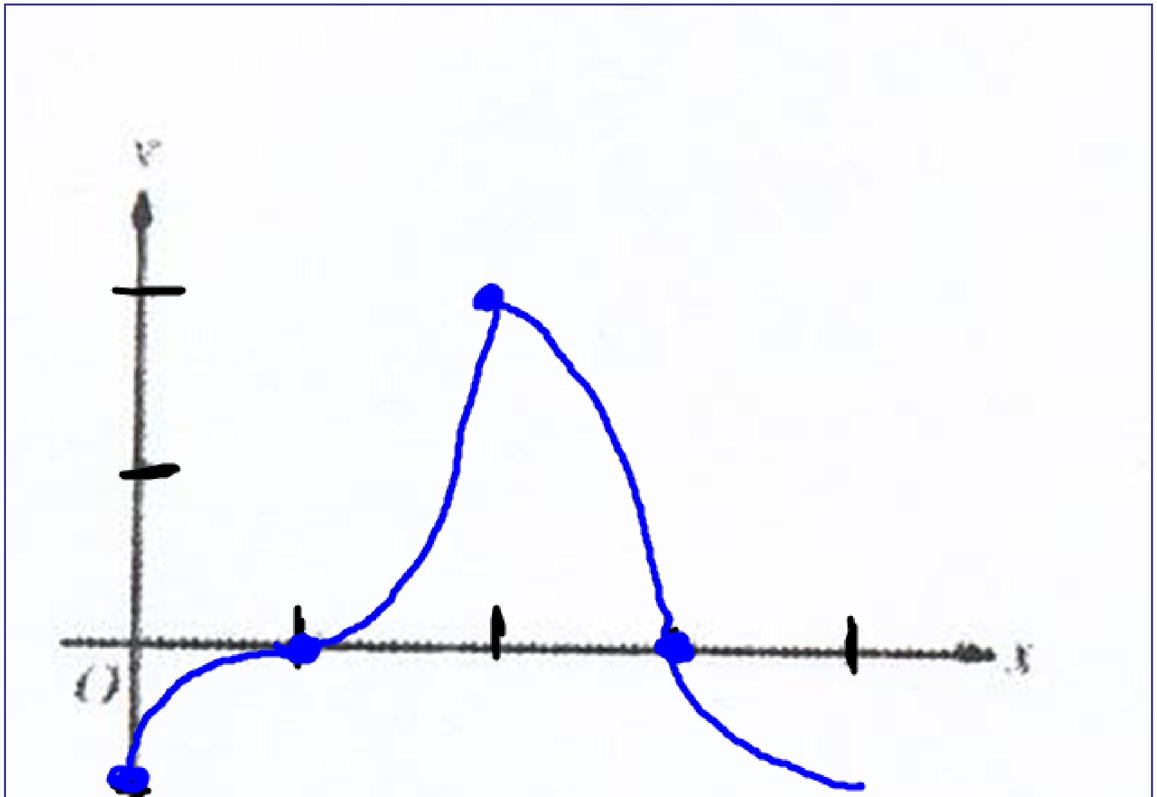


12
11
10
9
8
7
6
5
4
3
2
1
-12 -11 -10 -9 -8 -7 -6 -5 -4 -3 -2 -1 0 1 2 3 4 5 6 7 8 9 10 11 12
-1
-2
-3
-4
-5
-6
-7
-8
-9
-10
-11
-12

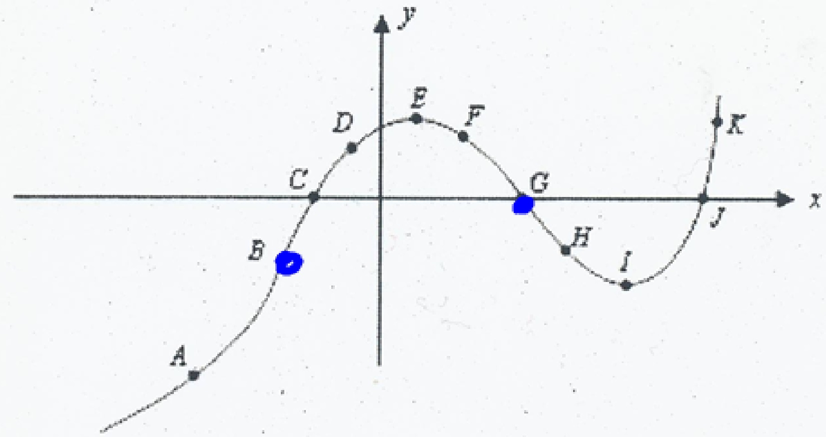
Continuous

Question 4

x	0	$0 < x < 1$	1	$1 < x < 2$	2	$2 < x < 3$	3	$3 < x < 4$
$f(x)$	-1	Negative	0	Positive	2	Positive	0	Negative
$f'(x)$	4	Positive	0	Positive	DNE	Negative	-3	Negative
$f''(x)$	-2	Negative	0	Positive	DNE	Negative	0	Positive



Fill in the blanks using +, -, or 0.



Point	$f(x)$	$f'(x)$	$f''(x)$
A	-	+	+
B	-	+	0
C	0	+	-
D	+	+	-
E	+	0	-
F	+	-	-
G	0	-	0
H	-	-	+
I	-	0	+
J	0	+	+
K	+	+	+

Assignment
Graphing Handout
4.3 Concavity and Points of Inflection
#3 i) ii)

Extra Practice Graphs of f'