4.2 The Mean Value Theorem and Rolle's Theorem

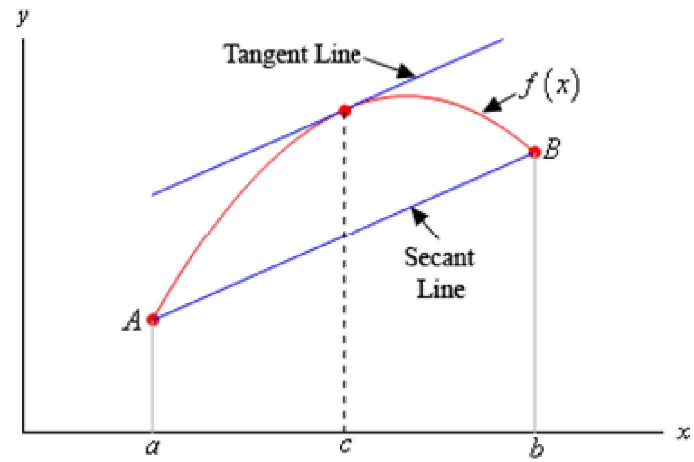
Mean Value Theorem

If *y=f(x)* is continuous at every point of a closed interval [a,b] and differentiable at every point of its interior (a,b), then there is at least one point "c" in (a,b) such that:

$$f'(x) = \frac{f(b)-f(a)}{b-a}.$$

slope tangent

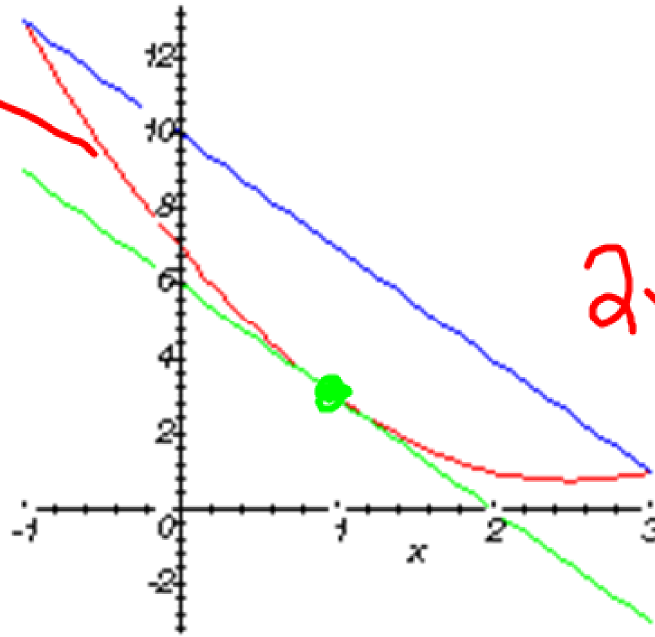
slope secant



we know f(x) diff

Ex.1 Find the value for *x* that satisfies the mean value theorem for $f(x) = x^2 - 5x + 7,\ -1 \le x \le 3$

f(x)



$$f'(x) = \frac{f(b) - f(a)}{b - a}$$

$$2x - 5 = \frac{f(3) - f(-1)}{3 + 1}$$

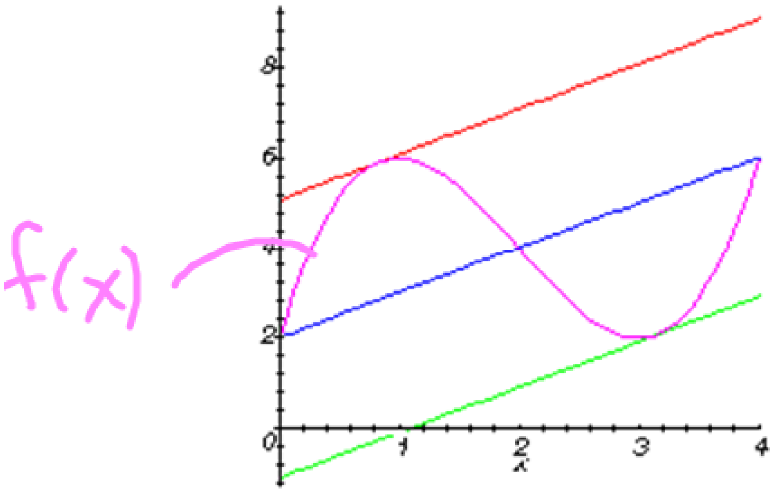
$$2x - 5 = \frac{1 - 13}{4}$$

$$2x - 5 = -3$$

$$2x = 2$$

$$\boxed{x = 1}$$

Ex.2 Find the value(s) for *x* such that satisfies the mean value theorem for $f(x) = x^3 - 6x^2 + 9x + 2,\ 0 \le x \le 4$

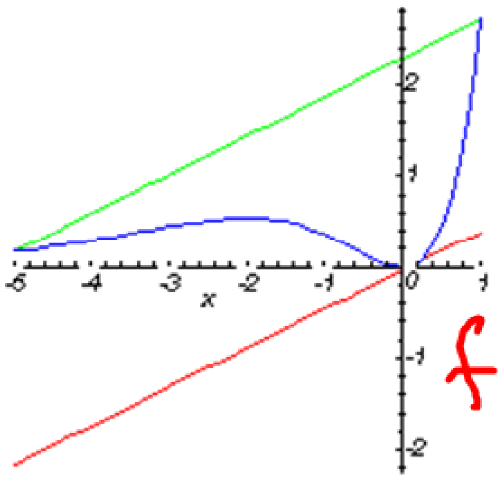


$$3x^2 - 12x + 9 = \frac{f(4) - f(0)}{4 - 0}$$

$$3x^2 - 12x + 9 = 1$$

$x = .845$ or $x = 3.155$

Ex.3 Find the value(s) for *x* such that satisfies the mean value theorem for $f(x) = x^2e^x,\ -5 \le x \le 1$



$f'(x) = x^2e^x + e^x \cdot 2x$

$f' = xe^x(x+2)$

$xe^x(x+2) = \dfrac{f(1) - f(-5)}{6}$

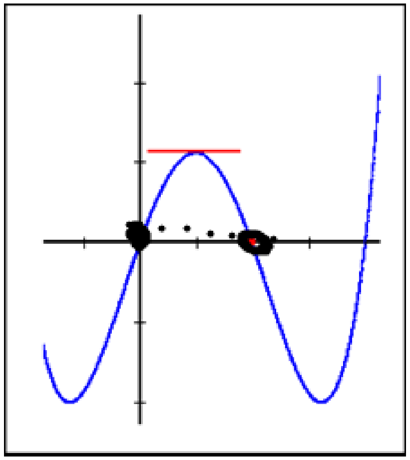
$x = .166$

Rolle's Theorem

Is just an extension of the Mean Value Theorem

If *f(x)* is continuous on a closed interval *[a,b]* and differentiable on the open interval *(a,b)*, then:

If $f(a) = f(b)$ then there is at least one c in (a,b) such that $f'(c) = 0$.



Ex.4 Given the function below, show that **Rolle's Theorem** applies on the interval [-4,0] and find all values of c that satisfy the theorem.

x

$$f(x) = x^2 + 4x - 5$$

Rolle Thm

$f(-4) = (-4)^2 + 4(-4) - 5 = -5$

$f(0) = (0)^2 + 4(0) - 5 = -5$

Rolle Thm applies

$f' = 0$

$2x + 4 = 0$

$2x = -4$

$x = -2$

Ex.5 Given the function below, show that **Rolle's Theorem** applies and find all values of c that satisfy the theorem. (Graphing calculator required.)

$$f(x)=\frac{x^3}{3}-\frac{x^2}{2}-2x+2$$

$f' = x^2 - x - 2$

$x^2 - x - 2 = 0$

$(x-2)(x+1) = 0$

$x = 2$ or $x = -1$

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2008 SCORING GUIDELINES

Question 2

t (hours)	0	1	3	4	7	8	9
$L(t)$ (people)	120	156	176	126	150	80	0

Concert tickets went on sale at noon $(t = 0)$ and were sold out within 9 hours. The number of people waiting in line to purchase tickets at time t is modeled by a twice-differentiable function L for $0 \le t \le 9$. Values of $L(t)$ at various times t are shown in the table above.

(a) Use the data in the table to estimate the rate at which the number of people waiting in line was changing at 5:30 P.M. $(t = 5.5)$. Show the computations that lead to your answer. Indicate units of measure.

(b) Use a trapezoidal sum with three subintervals to estimate the average number of people waiting in line during the first 4 hours that tickets were on sale.

(c) For $0 \le t \le 9$, what is the fewest number of times at which $L'(t)$ must equal 0 ? Give a reason for your answer.

(d) The rate at which tickets were sold for $0 \le t \le 9$ is modeled by $r(t) = 550te^{-t/2}$ tickets per hour. Based on the model, how many tickets were sold by 3 P.M. $(t = 3)$, to the nearest whole number?

(c) L is differentiable on $[0, 9]$ so the Mean Value Theorem implies $L'(t) > 0$ for some t in $(1, 3)$ and some t in $(4, 7)$. Similarly, $L'(t) < 0$ for some t in $(3, 4)$ and some t in $(7, 8)$. Then, since L' is continuous on $[0, 9]$, the Intermediate Value Theorem implies that $L'(t) = 0$ for at least three values of t in $[0, 9]$.	$3 : \begin{cases} 1 : \text{considers change in sign of } L' \\ 1 : \text{analysis} \\ 1 : \text{conclusion} \end{cases}$

Assignment
Handout

2 ( )

$$f(x) = \frac{x^2 - 2x - 3}{x+2} \qquad [-1, 3]$$

$$f(-1) = \frac{(-1)^2 - 2(-1) - 3}{-1+2} = 0$$

$$f(3) = \frac{9 - 6 - 3}{3-1} = 0$$

$$f'(x) = \frac{(x+2)(2x-2)-(x^2-2x-3)(1)}{(x+2)^2}$$

$$0 = 2x^2+2x-4-x^2+2x+3$$

$$0 = x^2+4x-1$$

$$x = \frac{-4 \pm \sqrt{(4)^2-4(1)(-1)}}{2(1)} = \frac{-4 \pm \sqrt{20}}{2}$$

$$x = \frac{-4 \pm 2\sqrt{5}}{2}$$

$$x = -2 \pm \sqrt{5}$$

$$x = -2 + \sqrt{5}$$

4.2 Increasing and Decreasing Intervals

The First Derivative Test

A function is said to be **increasing** if the graph is **"going uphill".**

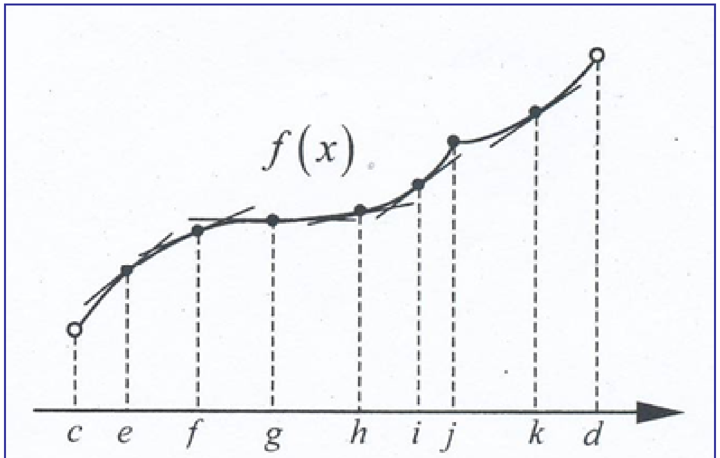


Figure 3

What do we notice about the slopes of the tangent lines for a function that is increasing? +

If $f'(x) > 0$ for all x in an open interval, then $f(x)$ is increasing on this open interval.

A function is said to be **decreasing** if the graph is **"going downhill".**

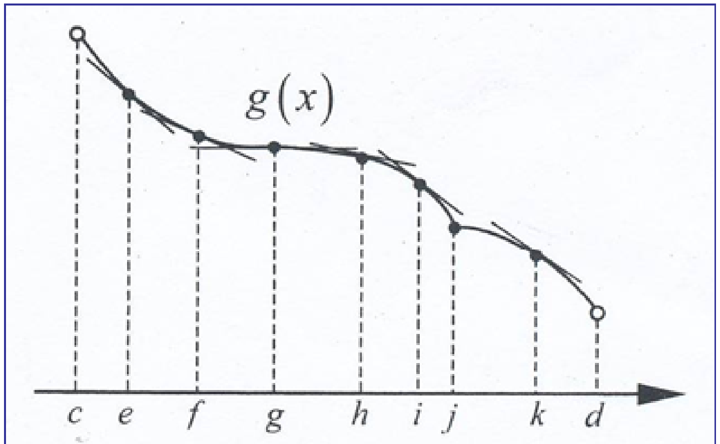


Figure 4

What do we notice about the slopes of the tangent lines for a function that is decreasing? —

If $f'(x) < 0$ for all x in an open interval,
then $f(x)$ is decreasing on this open interval.

2. **Critical Numbers**

Figures 1 through 4 illustrate that if a function $f(x)$ has a relative extremum at $x = c$, then either $f'(c) = 0$ or $f'(c)$ does not exist. If $x = c$ is in the domain of $f(x)$ and $f'(c) = 0$ or $f'(c)$ does not exist, then $x = c$ is said to be a **critical number** of $f(x)$. **Thus if $f(x)$ has a relative extremum at $x = c$, then $x = c$ is a critical number of $f(x)$.** Note that $x = c$ is a critical number in Figures 5 and 6 at right.

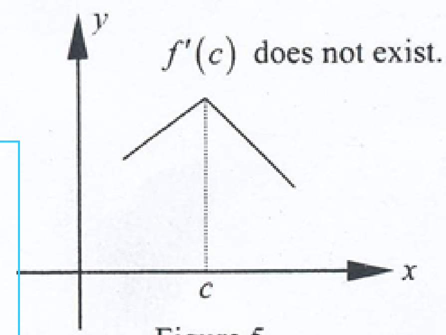


Figure 5

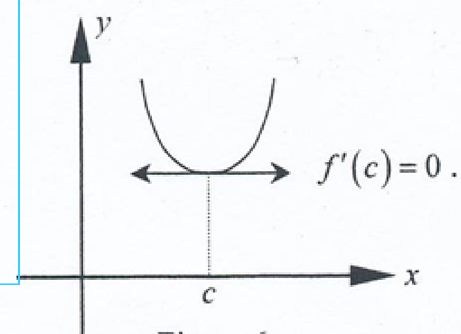


Figure 6

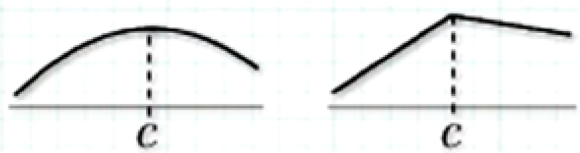
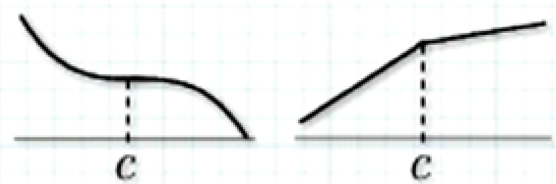
The First Derivative Test

If c is a critical number of a continuous function $f(x)$, then:

- $f(x)$ has a relative minimum at $x = c$ if $f'(x)$ switches signs from negative to positive at c.
- $f(x)$ has a relative maximum at $x = c$ if $f'(x)$ switches signs from positive to negative at c.

The First Derivative Test

Let c be an *isolated* critical number of f.

sign change in $f'(x)$ at c	picture	conclusion
$-$ | $+$		local minimum
$+$ | $-$		local maximum
none ($-$ | $-$ or $+$ | $+$)		not a local extremum

Ex.1 Find the open intervals for which the following function is increasing and decreasing. In addition find any relative extrema.

$$f(x)=\frac{1}{3}x^3+x^2-3x+1$$

$f' = x^2+2x-3$

$\underline{f'=0}$

$x^2+2x-3=0$

$(x+3)(x-1)=0$

$x=-3 \quad x=1$

$+$ 0 $-$ 0 $+$ f'

-4 $\;-3\;$ 0 $\;1\;$ 2

$\underline{\text{inc}}$

$(-\infty,-3)\cup(1,\infty)$

$\underline{\text{dec}}$

$(-3,1)$

Rel max

$f(-3) = \frac{1}{3}(-3)^3 + (-3)^2 - 3(-3) + 1$

$= -9 + 9 + 9 + 1$

$= 10$

$f(-3) = 10$

Rel min

$f(1) = \frac{1}{3}(1)^3 + (1)^2 - 3(1) + 1$

$= \frac{1}{3} - 1 = -2/3$

$(1, -2/3)$

Ex.2 Find the open intervals for which the following function is increasing and decreasing. In addition find any relative extrema.

$$f(x) = \frac{x^2}{x-2}$$

$x \neq 2$

$$f'(x) = \frac{(x-2)(2x) - x^2}{(x-2)^2}$$

$$f'(x) = \frac{x^2-4x}{(x-2)^2}$$

$f' = 0$

$x^2 - 4x = 0$

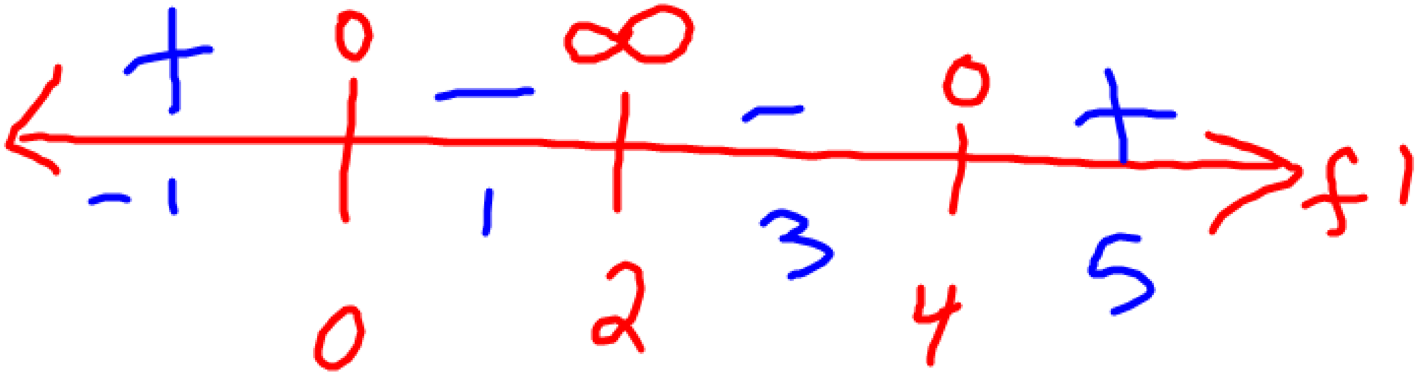
$x(x-4) = 0$

$x = 0$ $\quad$ $x = 4$

$f'\ \infty$

$(x-2)^2 = 0$

$x = 2$



Inc

$(-\infty, 0) \cup (4, \infty)$

Dec

$(0, 2) \cup (2, 4)$

Rel max

$f(0) = 0$

$(0, 0)$

Rel min

$f(4) = \dfrac{(4)^2}{4-2} = 8 \quad (4, 8)$

2003 MC No Calculator

15. Let f be the function with derivative given by $f'(x) = x^2 - \frac{2}{x}$. On which of the following intervals is f decreasing?

(A) $(-\infty, -1]$ only

(B) $(-\infty, 0)$

(C) $[-1, 0)$ only

(D) $(0, \sqrt[3]{2}]$

(E) $[\sqrt[3]{2}, \infty)$

$\underline{f' = 0}$

$x^2 - \frac{2}{x} = 0$

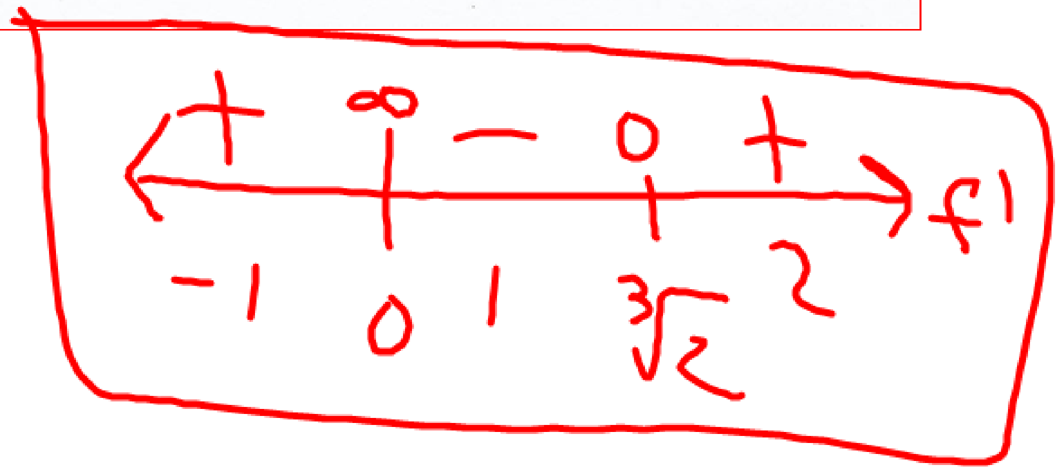
$x^2 = \frac{2}{x}$

$x^3 = 2$

$x = \sqrt[3]{2}$

$\underline{f' \ \infty}$

$\boxed{x = 0}$




JOB EVALUATION DAY

1997 MC No Calculator

22. What are all values of x for which the function f defined by $f(x) = (x^2 - 3)e^{-x}$ is increasing?

(A) There are no such values of x.
(B) $x < -1$ and $x > 3$
(C) $-3 < x < 1$
(D) $-1 < x < 3$
(E) All values of x

$$f' = (x^2-3)(-e^{-x}) + e^{-x}(2x)$$

$$= e^{-x}\left((x^2-3)(-1) + 2x\right)$$

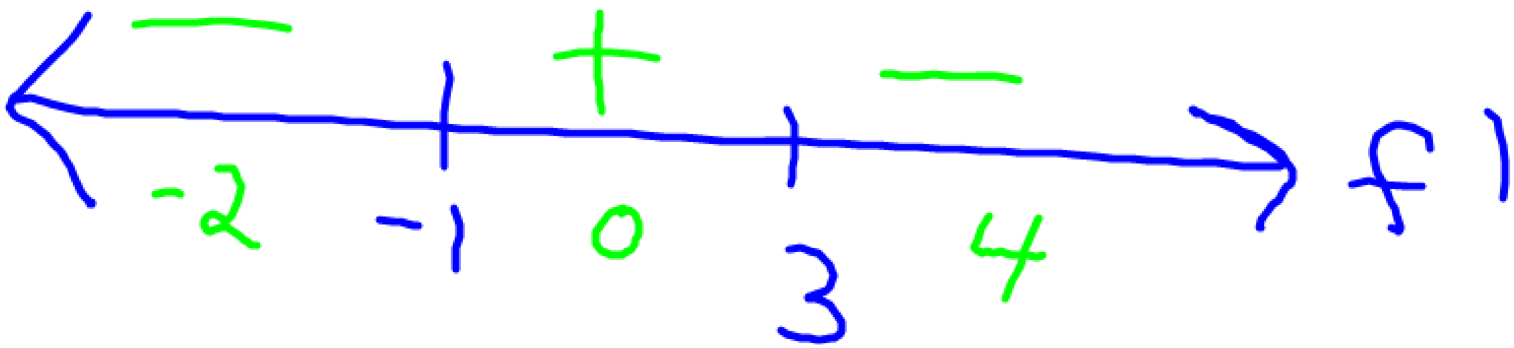
$$= e^{-x}\left(-x^2+3+2x\right)$$

$$f' = -e^{-x}(x^2-2x-3)$$

$$f' = \underbrace{-e^{-x}}_{-}(x-3)(x+1)$$

$x = 3$

$x = -1$




IT'S NOT WORTH IT, ROY! LET'S JUST GIVE HIM OUR NOSES & LET HIM GO!

1998 AP MC Question Calculator

80. The first derivative of the function f is given by $f'(x) = \frac{\cos^2 x}{x} - \frac{1}{5}$. How many critical values does f have on the open interval $(0, 10)$?

(A) One
(B) Three
(C) Four
(D) Five
(E) Seven


I TOLD YOU CIGARS WERE BAD FOR YOU!

Assignment

Calc 30 Text Page 235

#'s

2,3,4,5,7,8,10,11,13,15,16,17,18

4. The Interplay Between The Graphs Of $f(x)$ And $f'(x)$

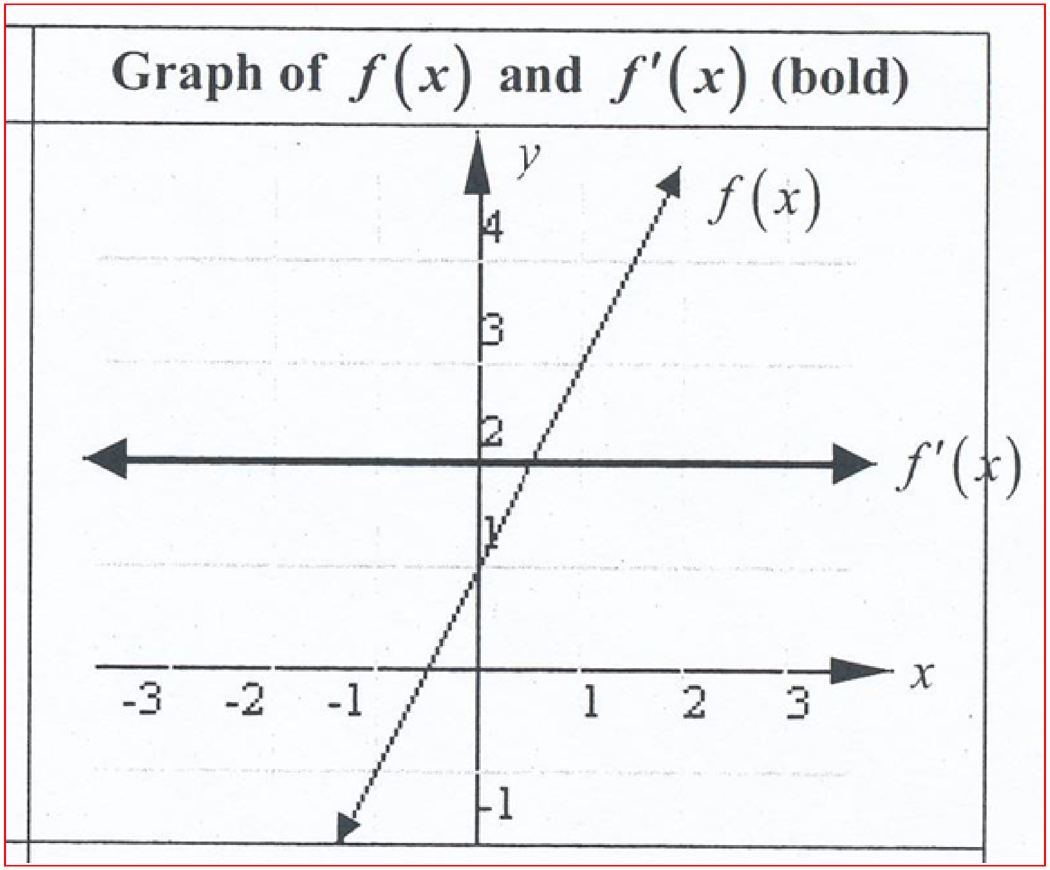
By studying the graph of a function, it should be possible to have a good idea as to the graph of its derivative and vice versa. In this section we have seen that if a function is increasing, its derivative is positive, and if a function is decreasing, its derivative is negative. Also, at a relative maximum or minimum point, the derivative is either 0 or does not exist. Study the table below that shows the graphs of several functions and their derivatives (in bold).

Surfer Derivative!

http://www.ies.co.jp/math/java/calc/doukan/doukan.html

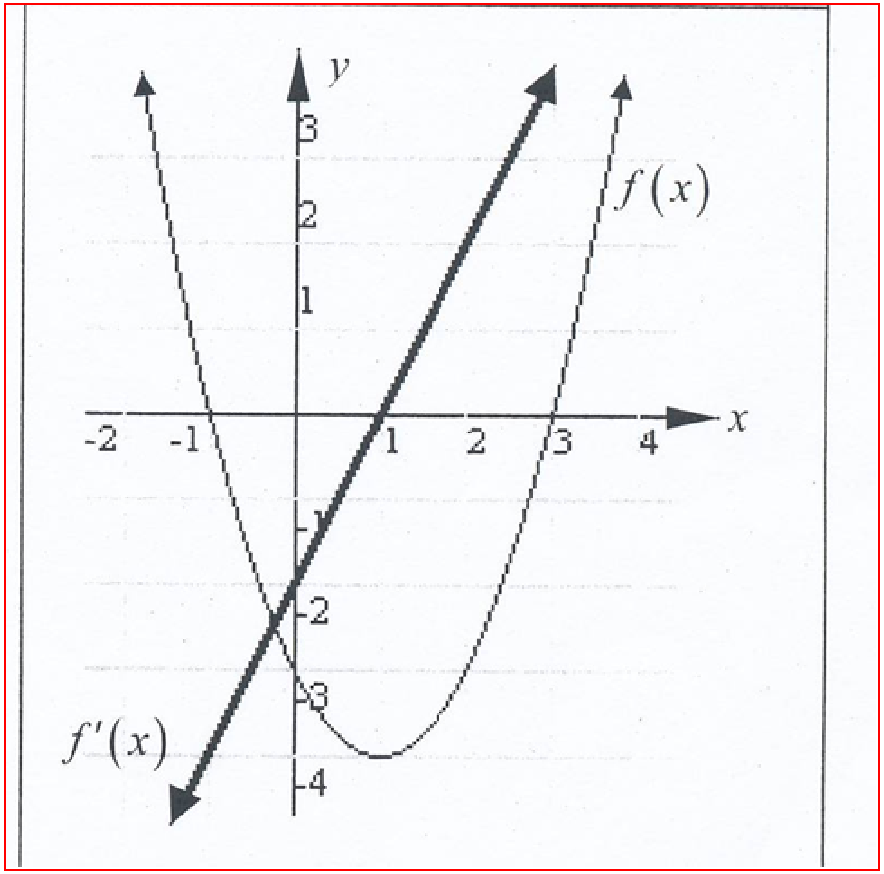
$$f(x) = 2x + 1$$

$f'(x) = 2$

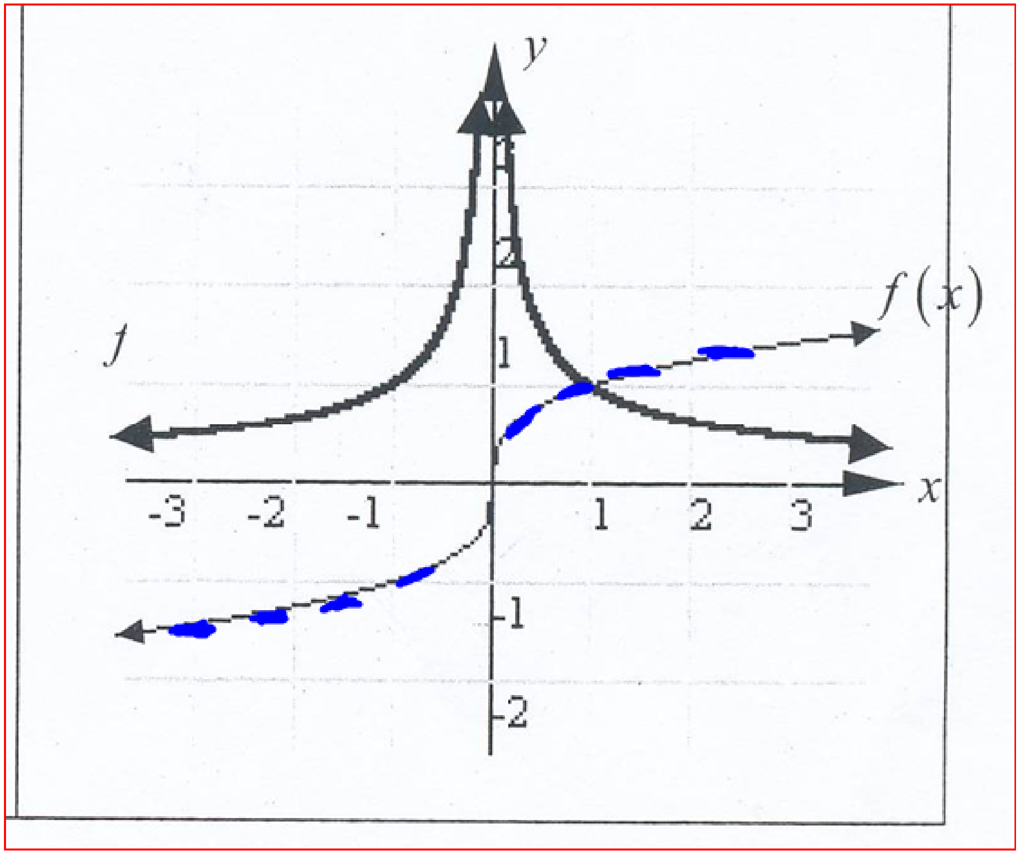


$$f(x) = x^2 - 2x - 3$$

$f' = 2x - 2$



$$f(x) = \sqrt[3]{x}$$



x = 1
rel max
f ' changes
from + to −
at x = 1.

30. Figure 4.108 is a graph of f'. For what values of x does f have a local maximum? A local minimum?

justify

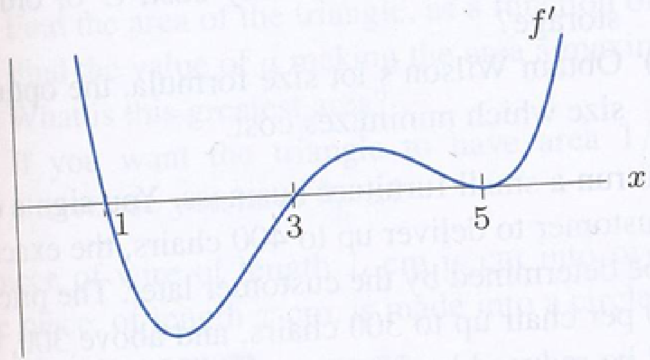


Figure 4.108: Graph of f' (not f)

x = 3
rel min
f ' changes
from − to +.

31. On the graph of f' in Figure 4.109, indicate the x-values that are critical points of the function f itself. Are they local maxima, local minima, or neither?

CP's

x = −1.2 → rel min
x = 1

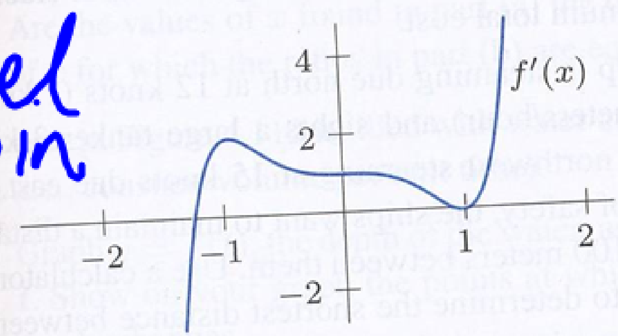
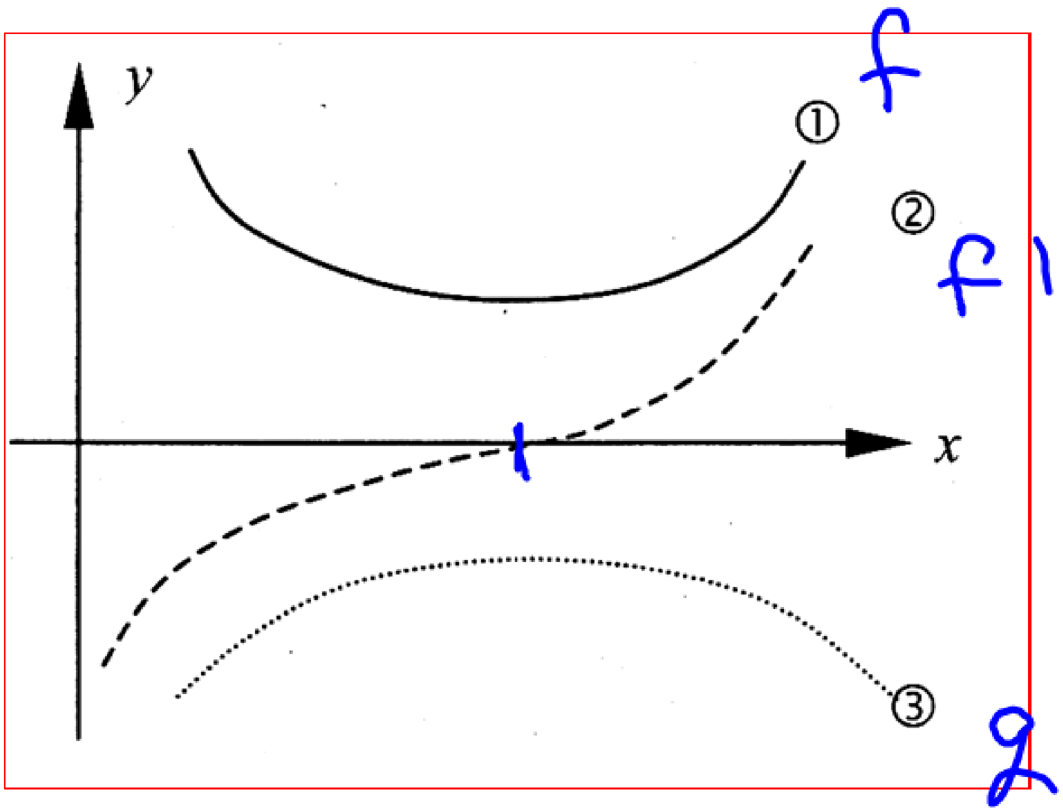


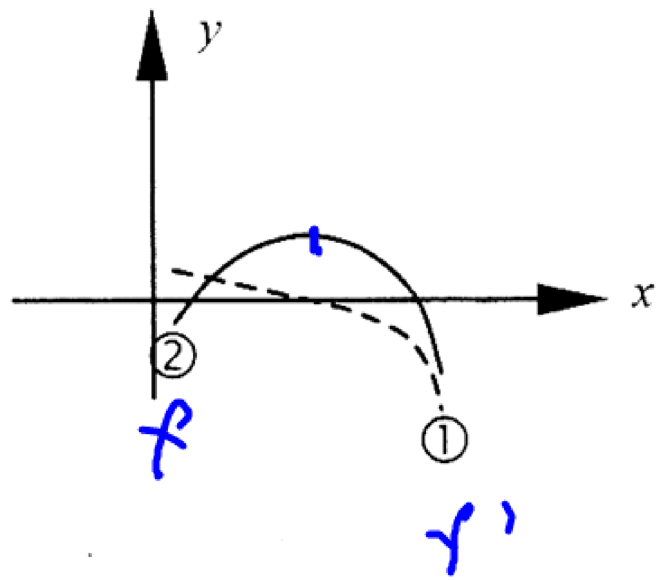
Figure 4.109: Graph of f' (not f)

Shown below are the graphs of $f(x)$, $f'(x)$, and $g(x)$. Which is which?



Your Turn #3

One of the graphs below is that of $f(x)$, the other is that of $f'(x)$. Which is which?



d) $x=5$ b/c f' Δ's from − to +.

e) $x=1$ max $x=5$ min

Example 7 Shown at right is the graph of $f'(x)$ on the interval $[0,6]$. Answer the following questions about $f(x)$.

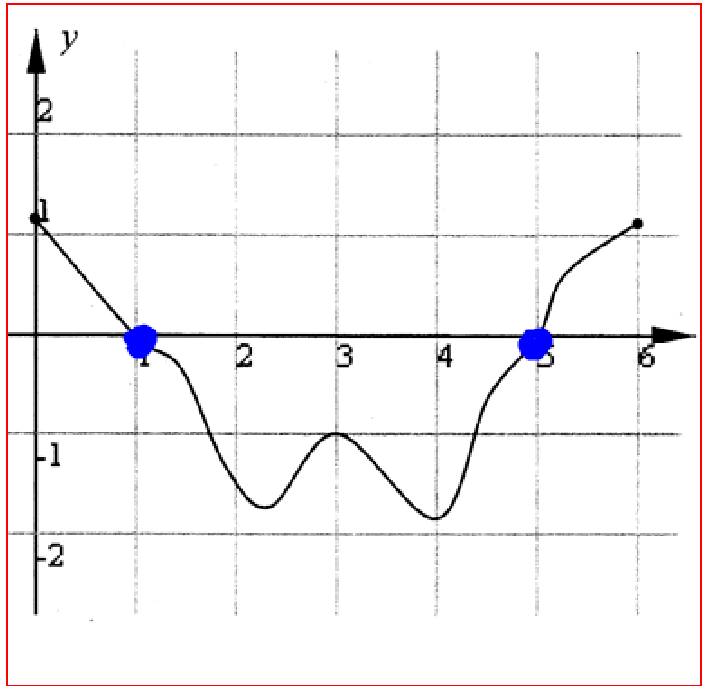
(a) On what open intervals is $f(x)$ increasing?

(b) On what open intervals is $f(x)$ decreasing?

(c) At what x-value is there a relative maximum?

(d) At what x-value is there a relative minimum?

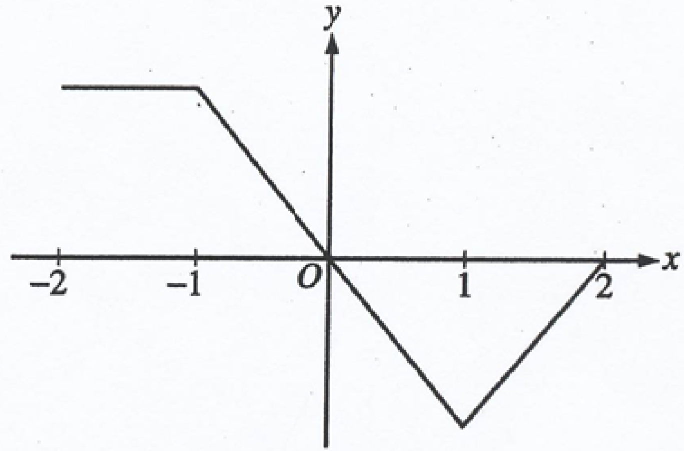
(e) At what x-value on the interval $[1,5]$ does $f(x)$ reach its maximum and minimum value?



a) $(0,1) \cup (5,6)$ $f'(x)>0$

b) $(1,5)$ $f'(x)<0$

c) $x=1$ b/c f' Δ's from + to −.



Graph of f'

7. The graph of f', the derivative of the function f, is shown above. Which of the following statements is true about f ?

(A) f is decreasing for $-1 \le x \le 1$.

(B) f is increasing for $-2 \le x \le 0$.

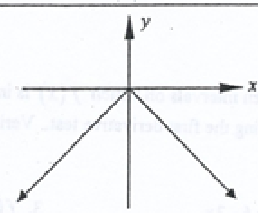
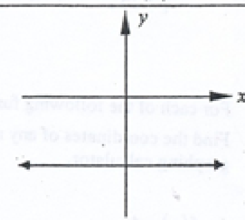
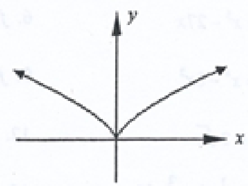
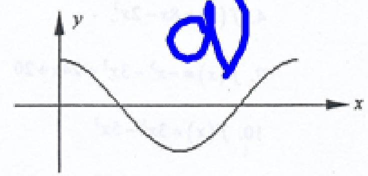
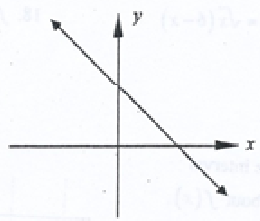
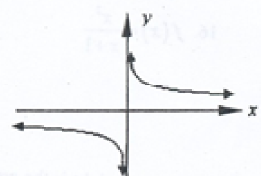
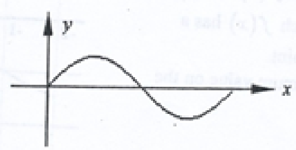
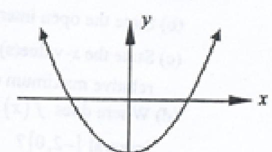
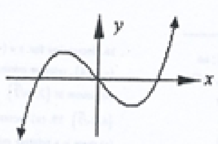
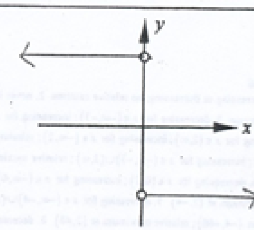
(C) f is increasing for $1 \le x \le 2$.

(D) f has a local minimum at $x = 0$.

(E) f is not differentiable at $x = -1$ and $x = 1$.

Assignment
Calc 30 Text Page 236
#'s 19-20

20. Match each function with the graph of its derivative. The scales are not necessarily the same from one graph to the next.

$f(x)$	$f'(x)$	
(a) 	(i) 	c)
(b) 	(ii) 	d)
(c) 	(iii) 	b)
(d) 	(iv) 	e)
(e) 	(v) 	a)

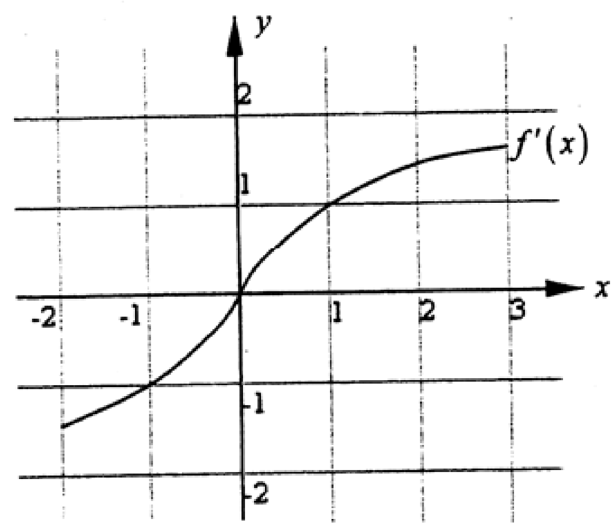
19. Shown at right is the graph of $f'(x)$ on the interval $[-2,3]$. Answer the following questions about $f(x)$. Explain your reasoning.

(a) State the open interval(s) on which $f(x)$ is increasing.

(b) State the open interval(s) on which $f(x)$ is decreasing.

(c) State the x-value(s), if any, at which $f(x)$ has a relative maximum or minimum point.

(d) Where does $f(x)$ reach its maximum value on the interval $[-2,0]$?



Curve Sketching With Maximums and Minimums.

We can use intervals of increase and decrease, along with relative maximums and minimums to draw a sketch of functions.

$1+4-9$

Find the intervals of increase and decrease of the following along with relative extrema to sketch the following function:

$$g(x) = x^4 - 4x^3 - 8x^2 - 1$$

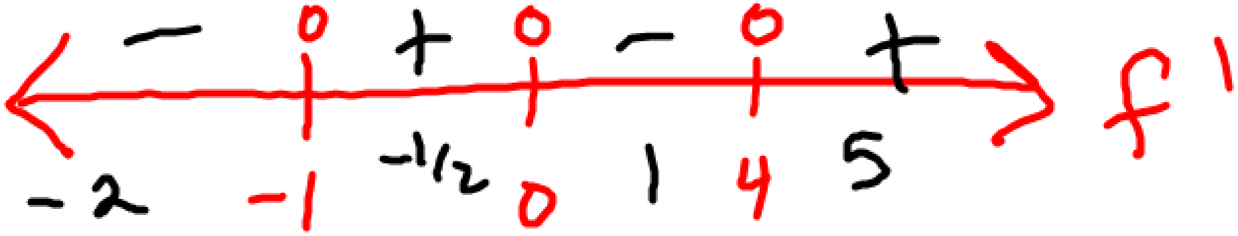
$$g'(x) = 4x^3 - 12x^2 - 16x$$

$$4x^3 - 12x^2 - 16x = 0$$

$$4x(x^2 - 3x - 4) = 0$$

$$4x(x-4)(x+1) = 0$$

$x=0$ or $x=4$ or $x=-1$



inc

$(-1, 0) \cup (4, \infty)$

dec

$(-\infty, -1) \cup (0, 4)$

Rel max

$g(0) = -1$

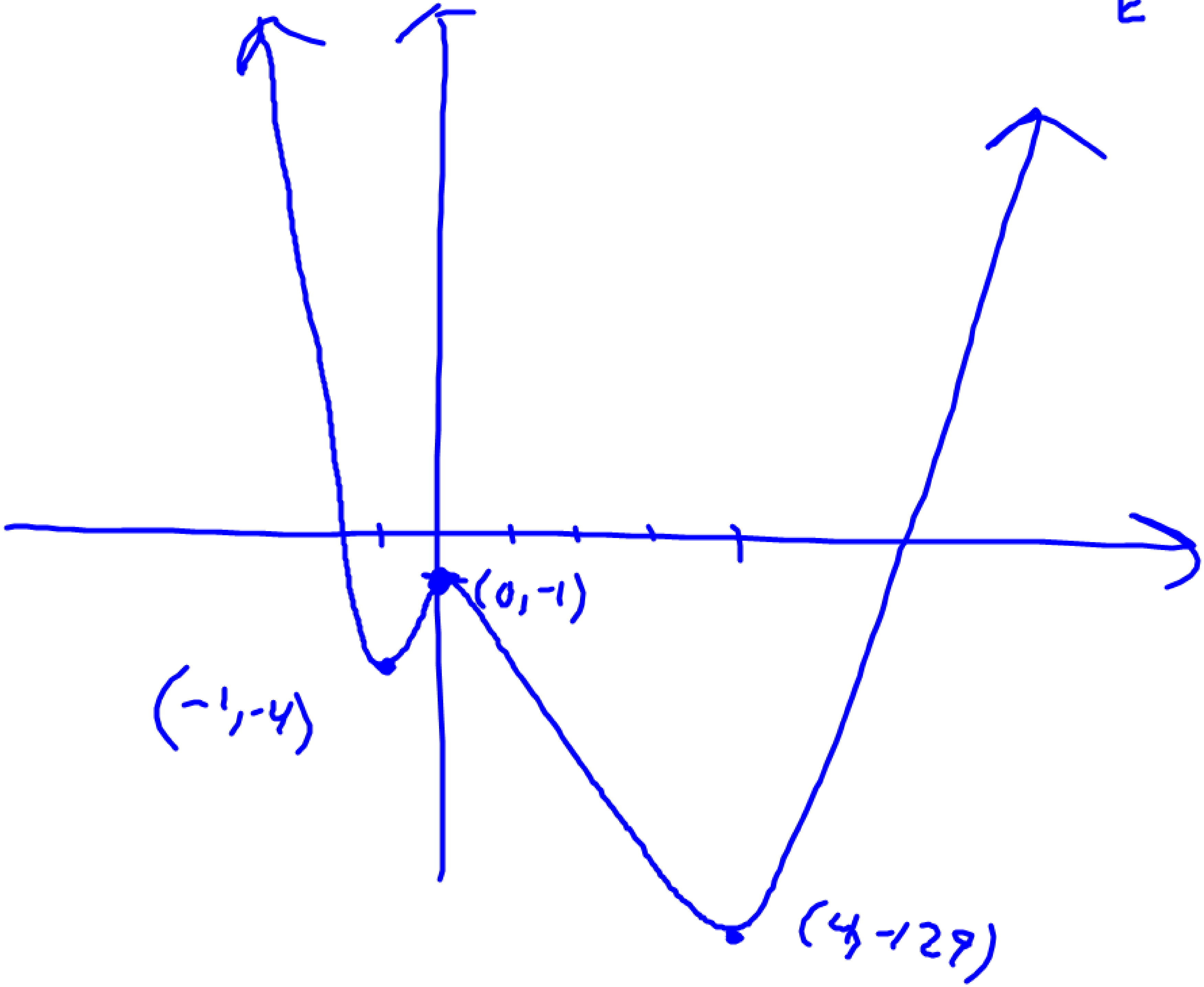
Rel min

$g(-1) = -4$

$g(4) = -129$

S

E



Assignment Handout
#2, #5