Unit 4 Application of Derivatives

4.1 Extreme Values of a Function

1. **Relative Extrema**

A function $f(x)$ is said to have **a local or relative maximum** value at $x = c$ if there exists an open interval containing c, on which $f(x)$ is defined, and $f(x) \leq f(c)$ for all values of x in the interval. Figures 1 and 2 below show a relative maximum at $x = c$. Note that in Figure 1, $f'(c) = 0$ while in Figure 2, $f'(c)$ does not exist.

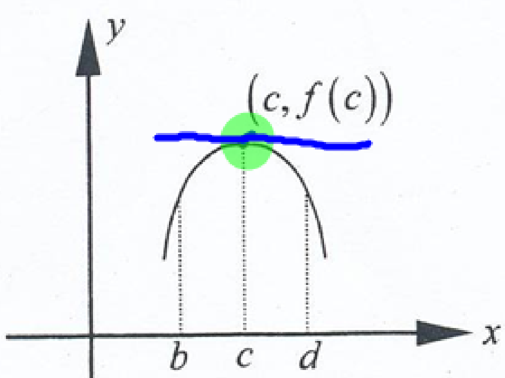


Figure 1

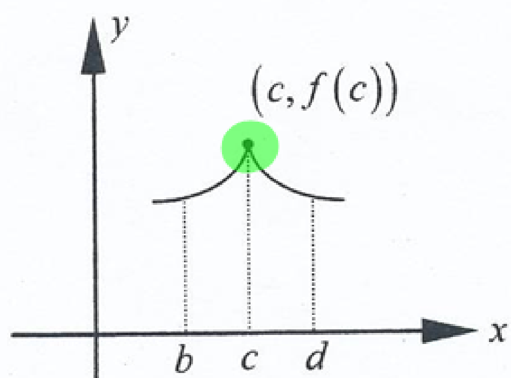


Figure 2

A function $f(x)$ is said to have **a local** or **relative minimum** value at $x = c$ if there exists an open interval containing c, on which $f(x)$ is defined, and $f(x) \geq f(c)$ for all values of x in the interval. Figures 3 and 4 below show a relative minimum at $x = c$. Note that in Figure 3, $f'(c) = 0$ while in Figure 2, $f'(c)$ does not exist.

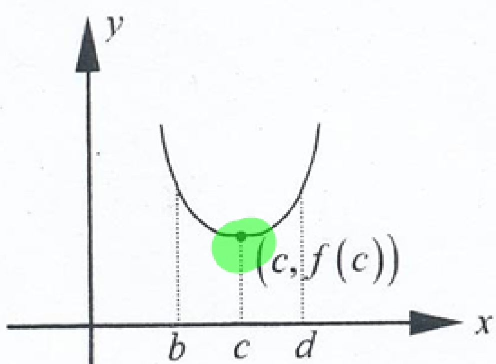


Figure 3

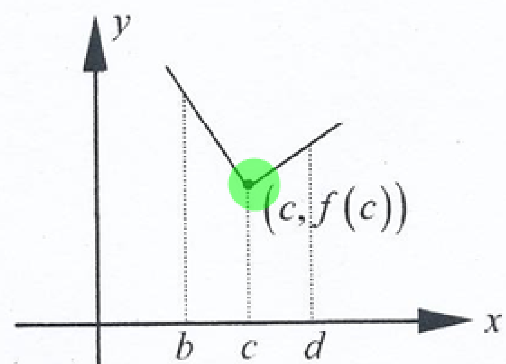


Figure 4

If a function $f(x)$ has either a relative maximum or a relative minimum at $x = c$, then $f(x)$ is said to have a **relative extremum** at $x = c$.

In Figures 1 and 2 the point with coordinates $\left(c, f(c)\right)$ is called a **relative maximum point** and $f(c)$ is a **relative maximum value**.

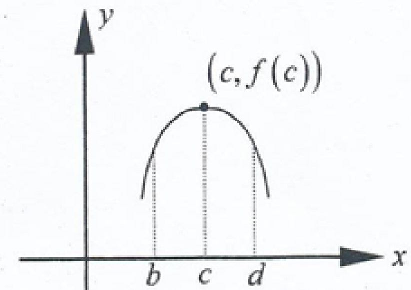


Figure 1

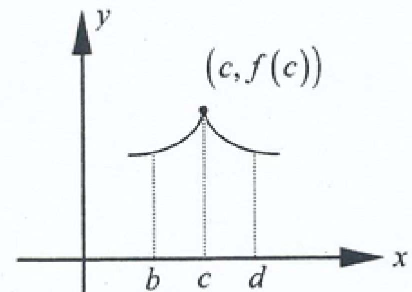


Figure 2

In Figures 3 and 4 the point with coordinates $\left(c, f(c)\right)$ is called a **relative minimum point** and $f(c)$ is a **relative minimum value**.

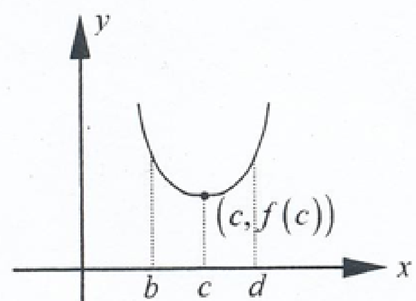


Figure 3

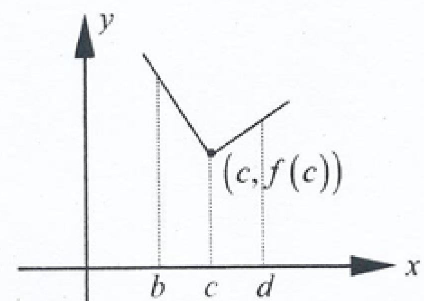


Figure 4

The plural forms of maximum, minimum, and extremum are maxima, minima, and extrema.

2. **Critical Numbers**

Figures 1 through 4 illustrate that if a function $f(x)$ has a relative extremum at $x = c$, then either $f'(c) = 0$ or $f'(c)$ does not exist. If $x = c$ is in the domain of $f(x)$ and $f'(c) = 0$ or $f'(c)$ does not exist, then $x = c$ is said to be a **critical number** of $f(x)$. **Thus if $f(x)$ has a relative extremum at $x = c$, then $x = c$ is a critical number of $f(x)$.** Note that $x = c$ is a critical number in Figures 5 and 6 at right.

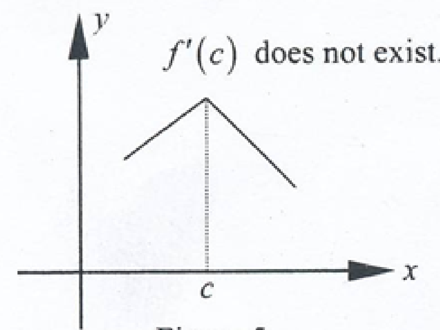


Figure 5

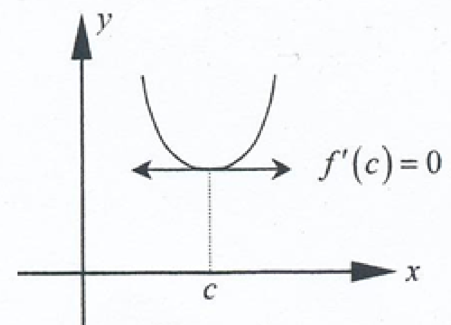


Figure 6

Ex.1 Find the **critical numbers** of the following function:

a) $f(x)=4x-6x^{\frac{2}{3}}$

$f' = 4 - 4x^{-1/3}$

$f' = \frac{4x^{1/3}}{x^{1/3}} - \frac{4}{x^{1/3}}$

$= \frac{4x^{1/3} - 4}{x^{1/3}}$

$\frac{f' = 0}{}$

$4x^{1/3} - 4 = 0$

$4x^{1/3} = 4$

$x^{1/3} = 1$

$\boxed{x = 1}$

$\frac{f' \infty}{x^{1/3} = 0}$

$x = 0$

b) $f(x)=\dfrac{2x}{x-3}$

$$f' = \frac{(x-3)(2) - 2x(1)}{(x-3)^2}$$

$$f' = \frac{2x-6-2x}{(x-3)^2}$$

$$f' = \frac{-6}{(x-3)^2}$$

$x \neq 3$

$$\frac{f' = 0}{-6 \neq 0}$$

$$\frac{f' \infty}{(x-3)^2 = 0}$$

$x = 3$

3. **Absolute Extrema**

Suppose $f(x)$ is a function that is defined on an interval I that contains $x = c$. Then $f(c)$ is an **absolute maximum** or **global maximum on *I*** if $f(c) \geq f(x)$ for all values of x in I. Similarly $f(c)$ is an **absolute minimum** or **global minimum on *I*** if $f(c) \leq f(x)$ for all values of x in I. If $f(c)$ is either an absolute maximum or an absolute minimum on I, it is also referred to as an **extreme value** or **absolute extremum on *I***. Figure 11 shows function $f(x)$ on the closed interval $[1,5]$. There are two critical numbers in

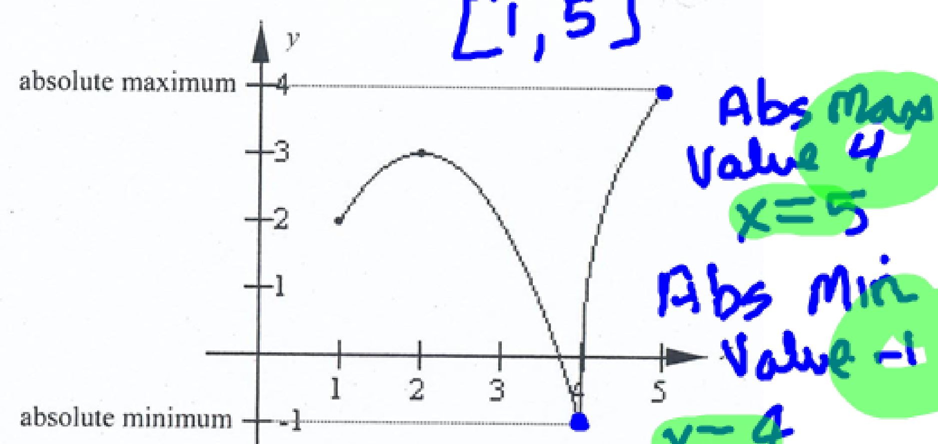


Figure 11

closed interval $[1,5]$. There are two critical numbers in
this interval—$x = 2$, since $f'(2) = 0$, and $x = 4$, since $f'(4)$ does not exist (a sharp corner). On this interval the function has an absolute maximum of 4 that occurs at $x = 5$, the right endpoint of the interval. The function has an absolute minimum of -1 that occurs at $x = 4$. Note, also, that $x = 4$ is the location of a relative minimum, and $x = 2$ is the location of a relative maximum. The value of the relative maximum is 3 since $f(2) = 3$. The extreme values or extrema of the function on this interval are 4 and -1.

Functions may or may not have an extreme value. The function $f(x) = x^3$, see Figure 7, has no global maximum or minimum since it contains no highest or lowest point. Likewise the function in Figure 12 has no global maximum or global minimum value on the interval (a,b) but has a global maximum of $f(a)$ and a global minimum of $f(b)$ on the interval $[a,b]$ as shown in Figure 13. The function in Figure 14 has a global maximum of $f(c)$ but no global minimum on the interval $[a,b)$. The discontinuous function in Figure 15 has a global minimum of $f(c)$ but has no global maximum on $[a,b)$.

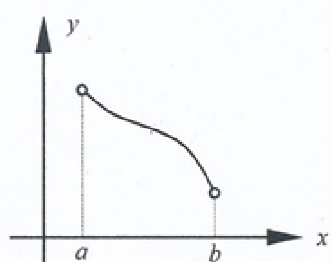


Figure 12

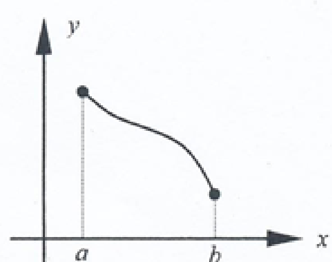


Figure 13

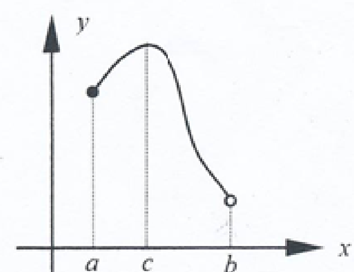


Figure 14

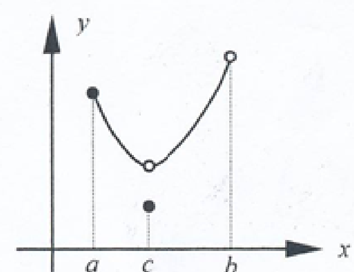


Figure 15

The Extreme Value Theorem

If *f(x)* is a continuous function on a closed interval *[a,b]*, then *f(x)* has both a global maximum and a global minimum on this interval.

Ex.2 Find the absolute maximum and absolute minimum values of the function on the given interval.

a) $f(x) = x^2 - 4x - 1, \ [1,4]$

$f'(x) = 2x - 4$

$f' = 0$

$2x - 4 = 0$

$2x = 4$

$x = 2$ ✓

$f(1) = -4$

$f(2) = -5$ Abs min

$f(4) = -1$ Abs max

$x \neq 1$

b) $y = \dfrac{(x+1)^2}{x-1}$, $[2,4]$

$$y' = \frac{(x-1)\cdot 2(x+1)^1 - (x+1)^2(1)}{(x-1)^2}$$

$$= \frac{(x+1)\big(2(x-1) - (x+1)\big)}{(x-1)^2}$$

$$= \frac{(x+1)(2x-2-x-1)}{(x-1)^2}$$

$$= \frac{(x+1)(x-3)}{(x-1)^2}$$

$\underline{y' = 0}$

$(x+1)(x-3) = 0$

$x = -1$ OR $x = 3$

$y(2) = 9$ Abs max

$y(3) = 8$ Abs min

$y(4) = \frac{25}{3}$

$\underline{y' \infty}$

$x = 1$

24. Let g be the function given by $g(x) = x^2 e^{kx}$, where k is a constant. For what value of k does g have a critical point at $x = \frac{2}{3}$?

(A) -3 (B) $-\frac{3}{2}$ (C) $-\frac{1}{3}$ (D) 0 (E) There is no such k.

$$g'(x) = x^2 \cdot e^{kx} \cdot k + e^{kx} \cdot 2x$$

$$g'(x) = k e^{kx} x^2 + 2x e^{kx}$$

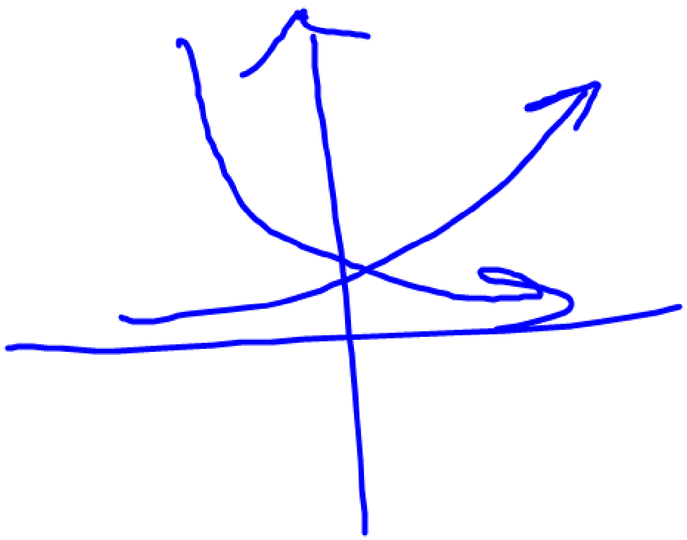
$$0 = k e^{k(2/3)} (2/3)^2 + 2(2/3) e^{(2/3)k}$$

$$0 = \frac{4}{9} k e^{2/3k} + \frac{4}{3} e^{2/3k}$$

$$0 = e^{2/3k} \left(4/9k + 4/3 \right)$$

~~$e^{2/3K} = 0$~~ or

$$\frac{4}{9}K + \frac{4}{3} = 0$$

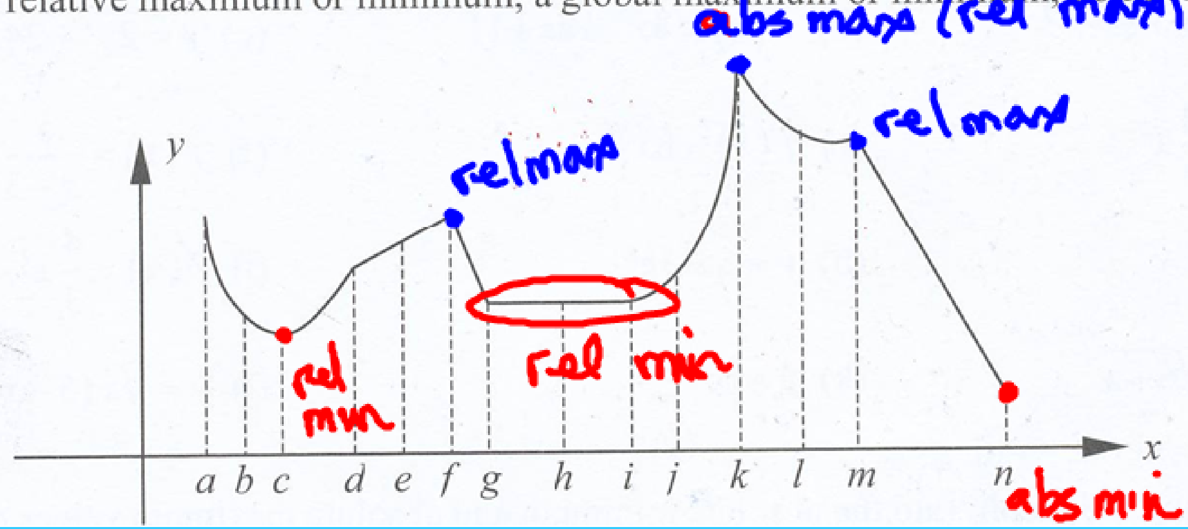


$$\frac{4}{9}K = -\frac{4}{3}$$

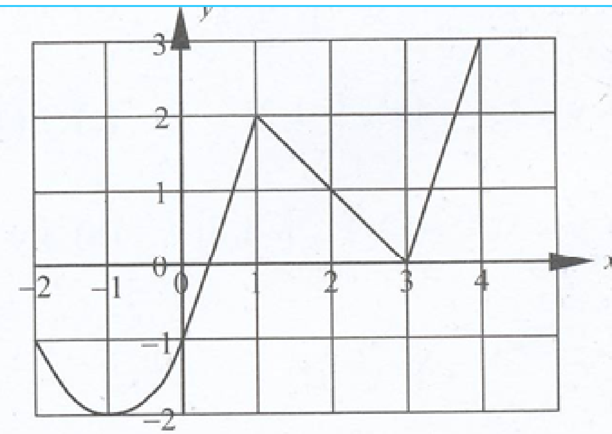
$$K = \left(-\frac{4}{3}\right)\frac{9}{4}$$

$$K = -3$$

Refer to the sketch of the function below and state whether, at each of the indicated x-values, the function has a relative maximum or minimum, a global maximum or minimum, both, or neither.



. Refer to the graph at right and give the absolute maximum and minimum values and the relative maximum and minimum values of the function on the interval $[-2,4]$.



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Page 185 #'s 45-48

Assignment

1. Sketch the graph of a function *that is continuous* on the interval $[-2,3]$ and has:

   (a) a global maximum of 3, a global minimum of 1, and no relative extrema.

   (b) a relative maximum value of 2 at $x=1$, a relative minimum value of 1 at $x=0$ and no other relative extrema; a global minimum of 0 at the right endpoint of the interval and a global maximum of 3 at the left endpoint of the interval.

   (c) a relative and global maximum value of 3 at $x=1$ and a relative and global minimum value of 1 at $x=-1$.

   (d) a critical number at $x=0$ but no relative maximum or minimum value.

2. Sketch the graph of a function on the interval $[-2,3]$ that has:

   (a) a global maximum but no relative maximum.

   (b) no global maximum and no global minimum.

   (c) a relative maximum and a relative minimum but no global maximum and no global minimum.

In Exercises 45–48, match the table with a graph of $f(x)$.

45.

x	$f'(x)$
a	0
b	0
c	5

46.

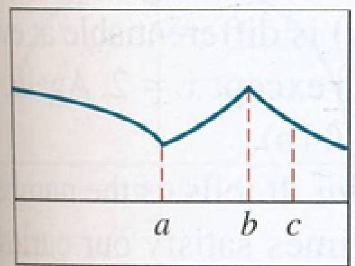
x	$f'(x)$
a	0
b	0
c	-5

47.

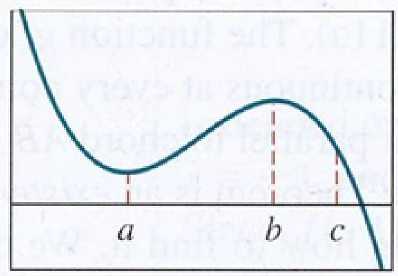
x	$f'(x)$
a	does not exist
b	0
c	-2

48.

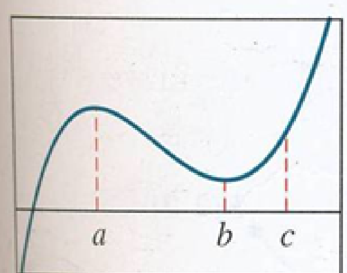
x	$f'(x)$
a	does not exist
b	does not exist
c	-1.7



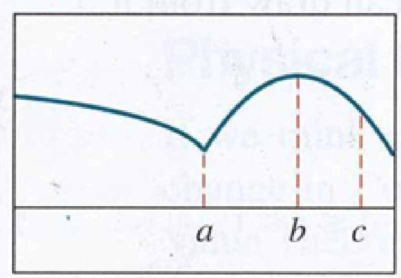
(a)



(b)



(c)



(d)