# Unit #4
# Derivatives of Inverses

# 4.1 Derivative of Inverses

A function is said to be one to one if for every x there is one and only one y.

Every function that is one to one has inverse function that is one to one.

We denote an inverse as $f^{-1}(x)$.

$\frac{1}{f(x)}$

Recall to find an inverse from Pre-Calc 30 we switch x and y and solve for y.

Find the inverse of $f(x) = 3x - 4$

$y = 3x - 4$

$x = 3y - 4$

$x + 4 = 3y$

$\frac{x+4}{3} = y$

If a function is continuous, its inverse is continuous.

If $f(x)$ is differentiable, then $f^{-1}(x)$ is also differentiable.

Also another fact about inverses is that if we compose a function with its inverse we will always end up with *x*.

$$f\left(f^{-1}(x)\right) = x$$

Demonstrate using previous example.

$$f(x) = y = 3x - 4 \qquad f^{-1}(x) = y = \frac{x+4}{3}$$

$$f(f^{-1}(x)) = x$$

$$f\left(\frac{x+4}{3}\right) = \cancel{3}\left(\frac{x+4}{\cancel{3}}\right) - 4$$

$$= x + 4 - 4$$

$$= x$$

Lets develop a formula for the derivative of an inverse starting with:

$$f\left(f^{-1}(x)\right) = x$$

$$\underbrace{f'\left(f^{-1}(x)\right)}\cdot\left(f^{-1}(x)\right)' = 1$$

$$\left(\underbrace{f^{-1}(x)}\right)' = \frac{1}{f'\left(f^{-1}(x)\right)}$$

$g(x)$ is inverse of $f(x)$

$$\left(g(x)\right)' = \frac{1}{f'\left(g(x)\right)}$$

This formula really says that the derivative of a function $f(x)$ at $(x, y)$ is the reciprocal of the derivative of $f^{-1}(x)$ at $(y, x)$

$f(2) = 1$ $\quad$ $f'(2) = \frac{1}{4}$

$g(1) = 2$ $\quad$ $g'(1) = 4$

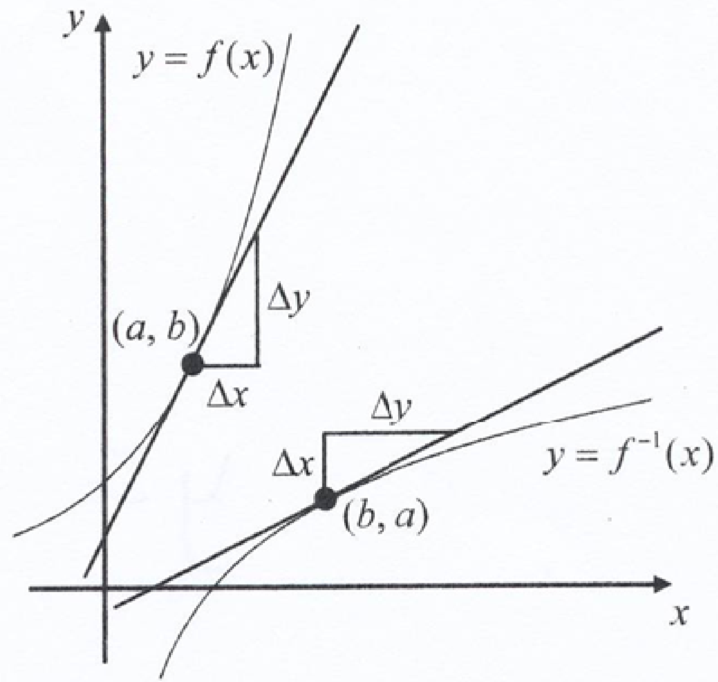


**Figure 3-2. Slope at (*a*, *b*) on the graph of *f* is reciprocal to slope at (*b*, *a*) on the graph of $f^{-1}$**

Example 1

Given $f(x) = \sqrt{x-4}$ and $\underbrace{g(x) = f^{-1}(x)}$, find $\underbrace{g'(1)}$.

$f(x) = 1$

$f(5) = 1$

$g(1) = 5$

$1 = \sqrt{x-4}$

$1 = x-4$

$\boxed{5 = x}$

$g'(x) = \dfrac{1}{f'(g(x))}$

$g'(1) = \dfrac{1}{f'(g(1))} = \dfrac{1}{f'(5)}$

$= \dfrac{1}{\left(\frac{1}{2}\right)} = 2$

$$f(x) = (x-4)^{1/2}$$

$$f'(x) = \frac{1}{2\sqrt{x-4}}$$

$$f'(5) = \frac{1}{2\sqrt{5-4}} = \frac{1}{2}$$

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Video Solution

Sometimes we are just asked to find the derivative of the inverse from a given function. Often we can use implicit differentiation to help us find the derivative.

## Example 1

If $f(\overset{y}{x}) = x^3 + 2x - 10$, find $\left(f^{-1}\right)'(x)$.

**Using implicit differentiation!**

$$x = 3y^3 + 2y - 10$$

$$1 = 9y^2\frac{dy}{dx} + 2\frac{dy}{dx}$$

$$1 = \frac{dy}{dx}\left(9y^2 + 2\right)$$

$$\frac{1}{(9y^2+2)} = \frac{dy}{dx}$$

$f(1) = 2$

27. Let $f$ be the function defined by $f(x) = x^3 + x$. If $g(x) = f^{-1}(x)$ and $g(2) = 1$, what is the value of $g'(2)$ ?

(A) $\frac{1}{13}$ (B) $\frac{1}{4}$ (C) $\frac{7}{4}$ (D) 4 (E) 13

$x = y^3 + y$

$1 = 3y^2 \frac{dy}{dx} + 1 \frac{dy}{dx}$

$\frac{1}{3y^2+1} = \frac{dy}{dx}$

$\frac{1}{3(1)^2+1} = \frac{dy}{dx} = \frac{1}{4}$

$f' = 3x^2 + 1$

$f'(1) = 3(1)^2 + 1$

$= 4$

$\frac{1}{4}$

Your Turn

Let $h(x) = 7 - x - 2x^5$ and let $f$ be the inverse function of $h$. Notice that $h(-1) = 10$.

$f'(10) =$ 

$h'(x) = -1 - 10x^4$

$h'(-1) = -1 - 10(-1)^4$

$= -11$

$f'(10) = -\frac{1}{11}$

Sometimes we need to use our knowledge of the relationship between the derivative of a function and the derivative of its inverse to solve problems.

28. Let $f$ be a differentiable function such that $f(3) = 15$, $f(6) = 3$, $f'(3) = -8$, and $f'(6) = -2$. The function $g$ is differentiable and $g(x) = f^{-1}(x)$ for all $x$. What is the value of $g'(3)$ ?

(A) $-\frac{1}{2}$

(B) $-\frac{1}{8}$

(C) $\frac{1}{6}$

(D) $\frac{1}{3}$

(E) The value of $g'(3)$ cannot be determined from the information given.

$= -\frac{1}{2}$

**Applying the formula:**

$$\left(f^{-1}\right)'(x) = \frac{1}{f'\left(f^{-1}(x)\right)}$$

$$\left(g^{-1}\right)'(3) = \frac{1}{f'\left(g^{-1}(3)\right)} = \frac{1}{f'(6)}$$

Let $f$ and $h$ be **inverse** functions.

The following table lists a few values of $f$, $h$, and $f'$.

| $x$ | $f(x)$ | $h(x)$ | $f'(x)$ |
|---|---|---|---|
| $-4$ | $2$ | $1$ | $\frac{1}{3}$ |
| $1$ | $-4$ | $-2$ | $4$ |

$f(1) = -4$

$f'(1) = 4$

$h'(-4) =$ $\frac{1}{4}$

Your Turn

Let $f$ and $h$ be **inverse** functions.

The following table lists a few values of $f$, $h$, and $h'$.

| $x$ | $f(x)$ | $h(x)$ | $h'(x)$ |
|---|---|---|---|
| $-1$ | $5$ | $8$ | $-2$ |
| $5$ | $1$ | $-1$ | $-\frac{2}{3}$ |

$$h(5) = -1$$

$$h'(5) = -\frac{3}{2}$$

$f'(-1) =$ $-2/3$

11. $\boxed{\text{C}}$ If $f$ and $f^{-1}$ are both differentiable for all $x$, with $f(3) = 5$ and $f'(3) = 7$, then which of the following must be a line tangent to the graph of $f^{-1}$?

(A) $y = 5 + 7(x - 3)$

(B) $y = \frac{1}{5} + \frac{1}{7}(x - 3)$

(C) $y = 3 + 7(x - 5)$

(D) $y = \frac{1}{3} + \frac{1}{7}(x - 5)$

(E) $y = 3 + \frac{1}{7}(x - 5)$

$f(3) = 5$

$f'(3) = 7$

$(5, 3)$ $m = \frac{1}{7}$

$y - 3 = \frac{1}{7}(x - 5)$

$g^{-1}(2) = 1$

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## Question 3

$g(1) = 2$

$(2, 1)$

$m = \frac{1}{5}$

| $x$ | $f(x)$ | $f'(x)$ | $g(x)$ | $g'(x)$ |
|---|---|---|---|---|
| 1 | 6 | 4 | 2 | 5 |
| 2 | 9 | 2 | 3 | 1 |
| 3 | 10 | $-4$ | 4 | 2 |
| 4 | $-1$ | 3 | 6 | 7 |

The functions $f$ and $g$ are differentiable for all real numbers, and $g$ is strictly increasing. The table above gives values of the functions and their first derivatives at selected values of $x$. The function $h$ is given by $h(x) = f(g(x)) - 6$.

(a) Explain why there must be a value $r$ for $1 < r < 3$ such that $h(r) = -5$.

(b) Explain why there must be a value $c$ for $1 < c < 3$ such that $h'(c) = -5$.

(c) Let $w$ be the function given by $w(x) = \int_1^{g(x)} f(t)\, dt$. Find the value of $w'(3)$.

(d) If $g^{-1}$ is the inverse function of $g$, write an equation for the line tangent to the graph of $y = g^{-1}(x)$ at $x = 2$.

$y - y_1 = m(x - x_1)$

$y - 1 = \frac{1}{5}(x - 2)$

Assignment Handout

## Derivatives of Inverses

1. Find the derivative of $f^{-1}$ at the point $(4,2)$ if $f(x) = x^3 + 2x - 8$.

2. Find the equation of the tangent line to the inverse at the given point

a) $f(x) = x^3 + 7x + 2$ @ $(10,1)$

b) $f(x) = x^5 + 3x^2 + 7x + 2$ @ $(13,1)$

c) $f(x) = e^{-2x} - 9x^3 + 4$ @ $(5,0)$

d) $f(x) = 7x + sin2x$ @ $(0,0)$

3. A function *f* and its derivative take on values shown in the table. If *g* is the inverse of *f*, find $g'(6)$.

| x | f(x) | f'(x) |
|---|---|---|
| 2 | 6 | 1/3 |
| 6 | 8 | 3/2 |

4. If $f(x) = x^3 + x$ and $g(x)$ is the inverse of $f(x)$, then find $g'(2)$, given $f(1) = 2$.

5. Let *f* and *g* differentiable. Given that $g(x) = f^{-1}(x)$, $f(1) = 3$, $f'(1) = -5$, find $g'(3)$.

6. Let *f* and *g* differentiable. Given that $g(x) = f^{-1}(x)$, $f(2) = 4$, $f(4) = -6$, $f'(2) = 7$, and $f'(4) = 11$. Find $g'(4)$.

7. Given that $g(x) = f^{-1}(x)$, and $g(-2) = 5$ and $f'(5) = \frac{-1}{2}$, then find $g'(-2)$.

## Answers

1. $\frac{1}{14}$ 2a) $y - 1 = \frac{1}{10}(x - 10)$ b) $y - 1 = \frac{1}{18}(x - 13)$ c) $y = \frac{-1}{2}(x - 5)$ d) $y = \frac{1}{9}x$

3. 3 4. $\frac{1}{4}$ 5. $\frac{-1}{5}$ 6. $\frac{1}{7}$ 7. -2