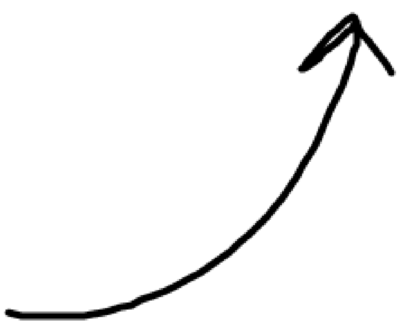
3.9 Derivatives of Exponential and Logarithmic Functions

Recall from Pre-Calc 30 **exponential functions** are of the form:

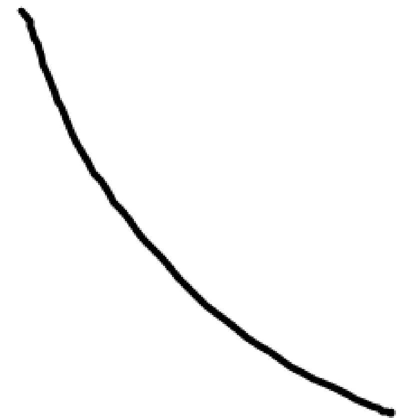
$y = b^x$

where b the base is any positve number except 1



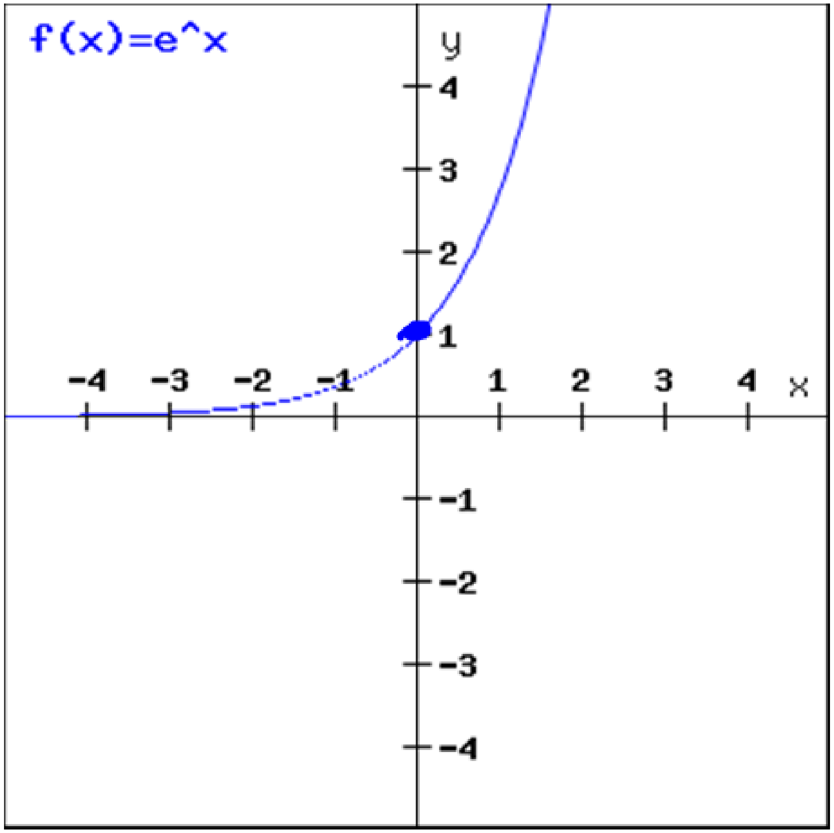
$$y = 2^x$$

$$y = \left(\frac{1}{3}\right)^x$$



In calculus we deal with a special number "e" which is a constant.

$$e = 2.718281828$$



When we differentiate exponential functions, we will be dealing with two kinds those involving base "**e**" and those involving base "a"

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Base "a" Exponential Derivatives

Ex.1 Differentiate the following:

$$a)\ y = 5^{3x}$$

$$y' = 5^{3x} \cdot 3 \cdot \ln 5$$

DONE

If $y = a^u$

then $\frac{dy}{dx} = a^u \cdot \frac{du}{dx} \cdot \ln a$

$$b)\ f(x) = 17^{4x^2}$$

$$f' = \left(17^{4x^2}\right)(8x)(\ln 17)$$

Base "e" Exponential Derivatives

Ex.1 Differentiate the following:

$a)\ y = e^{3x}$

$\log_e e = 1$

$y' = e^{3x} \cdot 3 \cdot \ln e = 1$

$= 3e^{3x}$

$\ln e = 1$

$\ln 1 = 0$

If $y = e^{u}$

then $\frac{dy}{dx} = e^{u} \cdot \frac{du}{dx}$

*

$$b)\ f(x) = e^{4x^2+3x}$$

$$f'(x) = e^{4x^2+3x} \cdot (8x+3)$$

$$c)\ y = \left(e^{4x+1}\right)^3$$

$$y' = 3\left(\underline{e^{4x+1}}\right)^2 \left(\underline{e^{4x+1}}\right)' \cdot 4$$

$$= 12\left(e^{4x+1}\right)^3$$

Good Math Humour!

A mathematician went insane and believed that he was the differentiation operator. His friends had him placed in a mental hospital until he got better. All day he would go around frightening the other patients by staring at them and saying "I differentiate you!"

One day he met a new patient; and true to form he stared at him and said "I differentiate you!", but for once, his victim's expression didn't change.

Surprised, the mathematician marshalled his energies, stared fiercely at the new patient and said loudly "I differentiate you!", but still the other man had no reaction. Finally, in frustration, the mathematician screamed out "I DIFFERENTIATE YOU!"

The new patient calmly looked up and said, "You can differentiate me all you like: I'm e to the x."

AP MC 1997 #76

If $f(x)=\dfrac{e^{2x}}{2x}$, then $f'(x)=$

a) 1

b) $\dfrac{e^{2x}(1-2x)}{2x^2}$

c) e^{2x}

d) $\dfrac{e^{2x}(2x+1)}{x^2}$

e) $\dfrac{e^{2x}(2x-1)}{2x^2}$

$$f'=\frac{2x(e^{2x}\cdot 2)-e^{2x}(2)}{(2x)^2}$$

$$=\frac{4xe^{2x}-2e^{2x}}{4x^2}$$

$$=\frac{\cancel{2}e^{2x}(2x-1)}{2\cancel{4}x^2}$$

AP Question 1997 MC Calculator Required

Let f be a function given by $f(x)=2e^{4x^2}$.

For what value of x is the slope of the line tangent to the graph of f at $(x, f(x))$ equal to 3.

$f' = 2e^{4x^2} \cdot 8x$

$f' = 16xe^{4x^2}$

$3 = 16xe^{4x^2}$

a) 0.168 b) 0.276 c) 0.318 d) 0.342 e) 0.551

AP Free Response 2006
#6

Question 6

The twice-differentiable function f is defined for all real numbers and satisfies the following conditions:

$$f(0) = 2, \quad f'(0) = -4, \text{ and } f''(0) = 3.$$

(a) The function g is given by $g(x) = e^{ax} + f(x)$ for all real numbers, where a is a constant. Find $g'(0)$ and $g''(0)$ in terms of a. Show the work that leads to your answers.

(b) The function h is given by $h(x) = \cos(kx)f(x)$ for all real numbers, where k is a constant. Find $h'(x)$ and write an equation for the line tangent to the graph of h at $x = 0$.

a) $g(x) = e^{ax} + f(x)$

$g'(x) = ae^{ax} + f'(x)$

$= a(1) + (-4)$

$= a - 4$

b) $f'' = a^2 e^{ax} + f''(x)$

$f''(0) = a^2 e^{a(0)} + f''(0)$

$= a^2 + 3$

b) $h(x) = (\cos kx)f(x)$

$(0, 2)$

$h(0) = \cos(k \cdot 0) \cdot f(0)$

$= (1)(\ 3\ )$

$= 2$

$m = (\cos kx)(f'(x)) + f(x)(-\sin kx)(k)$

$= (\cos(k \cdot 0))(f'(0)) + \cancel{f(0)(-\sin(k \cdot 0))k}$

$= (1)(-4)$

$= -4$

$y - 2 = -4(x - 0)$

AP Free Response 2011
#6

6. Let f be a function defined by $f(x) = \begin{cases} 1 - 2\sin x & \text{for } x \le 0 \\ e^{-4x} & \text{for } x > 0. \end{cases}$

(a) Show that f is continuous at $x = 0$.

(b) For $x \neq 0$, express $f'(x)$ as a piecewise-defined function. Find the value of x for which $f'(x) = -3$.

(c) Find the average value of f on the interval $[-1, 1]$.

a) $f(0) = 1 - 2\sin 0 = 1$

$\lim_{x \to 0} f(x)$

$$\lim_{x \to 0^+} e^{-4x} = 1$$

$$\lim_{x \to 0^-} 1 - 2\sin x = 1$$

$$\therefore \lim_{x \to 0} f(x) = 1$$

Since $f(0) = \lim_{x \to 0} f(x) = 1$ the function is continuous at $x = 0$

b) $f'(x) = \begin{cases} -2\cos x, & x \le 0 \\ -4e^{-4x}, & x > 0 \end{cases}$

$-2\cos x = -3$

$\cancel{\cos x = 3/2}$

$-4e^{-4x} = -3$

$e^{-4x} = 3/4$

$\ln e^{-4x} = \ln(3/4)$

$-4x = \ln(3/4)$

$x = -\frac{\ln(3/4)}{4}$

Assignment

Handout

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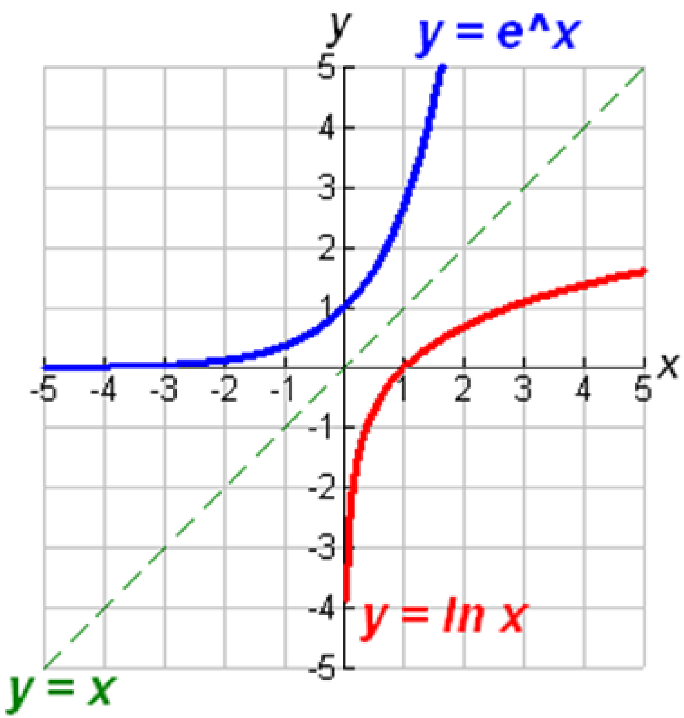
Recall from Pre-Calc 30 **that logarithmic functions are inverses of exponential functions.**



In calculus we must be able to differentiate common logarithms and natural logarithms.

$\log_2(x^2+3)$

$\ln(x^2+3)$

$\log_e$

Ex.1 Differentiate the following:

$a)\ y = \log_2 x^2$

$y' = \frac{1}{x^{\cancel{2}} \ln 2} \cdot 2\cancel{x} = \frac{2}{x \ln 2}$

Rule for Common Logs

$$\frac{d}{dx}\log_a u = \frac{1}{u \ln a}\frac{du}{dx}$$

$$b)\ y = 3\log_4\left(x^2-5\right)^3$$

$$y' = 3\left[\frac{1}{(x^2-5)^3 \cdot \ln 4} \cdot 3(x^2-5)^2 \cdot 2x\right]$$

$$= \frac{18x\,\cancel{(x^2-5)^2}}{(x^2-5)^{\cancel{3}\,2}\ln 4}$$

$$= \frac{18x}{(x^2-5)\ln 4}$$

$$c)\ y = \log_a a^{\sin x}$$

$y = \sin x$

$y' = \cos x$

Differentiating **Natural Logarithms** is much easier!

$$\frac{d}{dx}(\ln u) = \frac{1}{u}\frac{du}{dx}$$

Ex.2 Differentiate the following:

$$a)\ y = \ln 3x$$

$$y' = \frac{1}{3x} \cdot 3 = \frac{1}{x}$$

b) $y = \ln 4x^2 e^{2x}$

$$y' = \ln 4x^2 \cdot 2e^{2x} + e^{2x} \cdot \frac{1\;2}{\cancel{4}x^{\cancel{2}}} \cdot \cancel{8x}$$

$$= 2e^{2x} \ln 4x^2 + \frac{2}{x} e^{2x}$$

$$= 2e^{2x}\left(\ln 4x^2 + \frac{1}{x}\right)$$

c) $f(x)=\ln\left(\frac{x-4}{3x+1}\right)$

$$f' = \frac{1}{\left(\frac{x-4}{3x+1}\right)}\left[\frac{(3x+1)(1)-(x-4)(3)}{(3x+1)^2}\right]$$

$$= \left(\frac{\cancel{3x+1}}{x-4}\right)\left[\frac{\overset{13}{3x+1-3x+12}}{(3x+1)^{\cancel{2}}}\right]$$

$$= \frac{13}{(x-4)(3x+1)}$$

AP Question 2003 MC

9. If $f(x) = \ln\left(x + 4 + e^{-3x}\right)$, then $f'(0)$ is

(A) $-\frac{2}{5}$ (B) $\frac{1}{5}$ (C) $\frac{1}{4}$ (D) $\frac{2}{5}$ (E) nonexistent

$$f' = \frac{1}{x + 4 + e^{-3x}} \cdot \left(1 + e^{-3x}(-3)\right)$$

$$f'(0) = \frac{1}{0 + 4 + e^{-3(0)}} \cdot \left(1 + e^{-3(0)}(-3)\right)$$

$$= \left(\frac{1}{5}\right)\left(-2\right) = -2/5$$

Ex.3 Find dy/dx if:

$$y = x^{x}, x > 0$$

Note: We have a variable raised to a variable!

To differentiate we use logarithmic differentiation!

$\ln y = \ln x^{x}$

$\ln y = x \ln x$

$\frac{1}{y}\frac{dy}{dx} = x \cdot \frac{1}{x} + \ln x (1)$

$$\frac{1}{y}\frac{dy}{dx} = 1 + \ln x$$

$$\frac{dy}{dx} = y(1+\ln x)$$

$$= x^x(1+\ln x)$$

Question 6

Let f be the function given by $f(x) = \dfrac{\ln x}{x}$ for all $x > 0$. The derivative of f is given by

$f'(x) = \dfrac{1 - \ln x}{x^2}$.

(a) Write an equation for the line tangent to the graph of f at $x = e^2$.

(b) Find the x-coordinate of the critical point of f. Determine whether this point is a relative minimum, a relative maximum, or neither for the function f. Justify your answer.

(c) The graph of the function f has exactly one point of inflection. Find the x-coordinate of this point.

(d) Find $\lim\limits_{x \to 0^+} f(x)$.

a) $f(e^2) = \dfrac{\ln e^2}{e^2} = \dfrac{2 \ln e}{e^2} = \dfrac{2}{e^2}$

$\left(e^2, \dfrac{2}{e^2}\right)$

$$m = f'(e^2) = \frac{1 - \ln e^2}{(e^2)^2}$$

$$= \frac{1-2}{e^4} = -\frac{1}{e^4}$$

$$\left(e^2, \frac{2}{e^2}\right)$$

$$y - y_1 = m(x - x_1)$$

$$y - \frac{2}{e^2} = -\frac{1}{e^4}(x - e^2)$$

Assignment
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#’s7,8,11,12,17,18,19,21,22,
30,31,41,42

1. If $f(x) = 6x^{3/2}$, then $f''(16) =$

(A) 9/16
(B) 9/8
(C) 36
(D) 144
(E) 384

2. The slope of $9x - 4x \ln y = 3$ at $(1/3,1)$ is

(A) $9 - 4 \ln 3$
(B) 5
(C) 6
(D) 27/4
(E) $9 + 4 \ln 3$

10. If $g(x) = \csc x - \cot x$, then $g'(\pi/6) =$

(A) $4 - 2\sqrt{3}$

(B) $2 - 2\sqrt{3}$

(C) $2 - \sqrt{3}$
(D) 1
(E) $4 + \sqrt{3}$

11. If $s(x) = \sin^2 x$, then $s''(x) =$

(A) -2
(B) $-2 \cos x \sin x$
(C) $2 \sin x \cos x$
(D) $2 \cos^2 x - 2 \sin^2 x$
(E) 2