3.8 Inverses and Derivatives of Inverses

A function is said to be one to one if for every x there is one and only one y.

Every function that is one to one has inverse function that is one to one.

We denote an inverse as $f^{-1}(x)$. $\neq \frac{1}{f(x)}$

Recall to find an inverse from Pre-Calc 30 we switch x and y and solve for y.

$$y = 3x + 4$$

$$x = 3y + 4$$

$$x - 4 = 3y$$

$$\frac{1}{3}x - \frac{4}{3} = y$$

$$\frac{(x-4)}{3} = y$$

$$y = x^2 - 4$$

$$x = y^2 - 4$$

$$x + 4 = y^2$$

$$\pm\sqrt{x+4} = y$$

If a function is continuous, its inverse is continuous.

If $f(x)$ is differentiable, then $f^{-1}(x)$ is also differentiable.

Also another fact about inverss is that if we compose a function with its inverse we will always end up with *x*.

$$f\left(f^{-1}(x)\right) = x$$

Lets develop a formula for the derivative of an inverse starting with:

$$f\left(f^{-1}(x)\right) = x$$

$$f'\left(f^{-1}(x)\right) \frac{d}{dx} f^{-1}(x) = 1$$

$$\frac{d}{dx} f^{-1}(x) = \frac{1}{f'\left(f^{-1}(x)\right)}$$

Thus, the derivative of the inverse is the reciprocal of the derivative of the original function, but evaluated at the point $f^{-1}(x)$ instead of the point x.

Figure 3.28 shows $f(x)$ and $f^{-1}(x)$. Using Table 3.6, find

(a) (i) $f(2) = 4$ (ii) $f^{-1}(2) = 1$ (iii) $f'(2)$ 2.8 (iv) $(f^{-1})'(2)$ $\frac{1}{1.4}$

(b) The equation of the tangent lines at the points P and Q.

(c) What is the relationship between the two tangent lines?

Table 3.6

x	$f(x)$	$f'(x)$
0	1	0.7
1	2	1.4
2	4	2.8
3	8	5.5

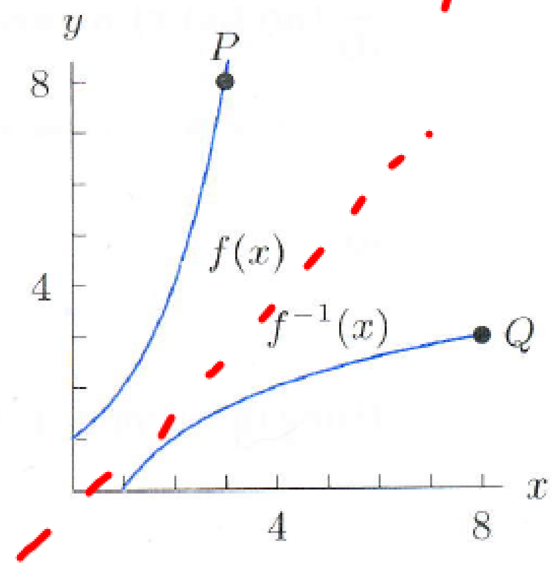

Figure 3.28

Solution:

(a) Reading from the table, we have

(i) $f(2) = 4$.

(ii) $f^{-1}(2) = 1$.

(iii) $f'(2) = 2.8$.

(iv) To find the derivative of the inverse function, we use

$$(f^{-1})'(2) = \frac{1}{f'(f^{-1}(2))} = \frac{1}{f'(1)} = \frac{1}{1.4} = 0.714.$$

Notice that the derivative of f^{-1} is the reciprocal of the derivative of f. However, the derivative of f^{-1} is evaluated at 2, while the derivative of f is evaluated at 1, where $f^{-1}(2) = 1$ and $f(1) = 2$.

(b) At the point P, we have $f(3) = 8$ and $f'(3) = 5.5$, so the equation of the tangent line at P is

$$y - 8 = 5.5(x - 3).$$

At the point Q, we have $f^{-1}(8) = 3$, so the slope at Q is

$$(f^{-1})'(8) = \frac{1}{f'(f^{-1}(8))} = \frac{1}{f'(3)} = \frac{1}{5.5}.$$

Thus, the equation of the tangent line at Q is

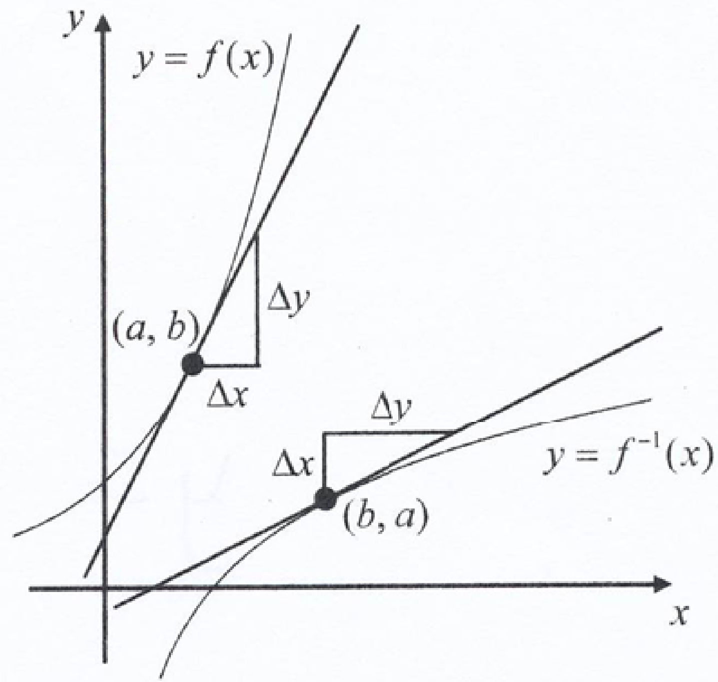

Figure 3-2. Slope at (*a*, *b*) on the graph of *f* is reciprocal to slope at (*b*, *a*) on the graph of f^{-1}

To demonstrate this look at the equation y = mx+b.

$m = \text{slope}$

$x = my + b$

$\frac{x}{m} - \frac{b}{m} = \cancel{m}y$

$\frac{1}{m}x - \frac{b}{m} = y$

$\frac{1}{m} = \text{slope}$


Inverse Video

Inverse Example

points

28. Let f be a differentiable function such that $f(3) = 15$, $f(6) = 3$, $f'(3) = -8$, and $f'(6) = -2$. The function g is differentiable and $g(x) = f^{-1}(x)$ for all x. What is the value of $g'(3)$?

(A) $-\frac{1}{2}$

(B) $-\frac{1}{8}$

(C) $\frac{1}{6}$

(D) $\frac{1}{3}$

(E) The value of $g'(3)$ cannot be determined from the information given.

inverse $(3, ?)$

function $(6, 3)$

Applying the formula:

$$\left(f^{-1}\right)'(x) = \frac{1}{f'\left(f^{-1}(x)\right)}$$

$$g'(x) = \frac{1}{f'(g(x))} = \frac{1}{f'(g(3))} = \frac{1}{f'(6)} \frac{1}{-2}$$

Example 1

If $f(x) = x^3 + 2x - 10$, find $\left(f^{-1}\right)'(x)$.

y

Using implicit differentiation!

$$x = y^3 + 2y - 10$$

$$1 = 3y^2 \frac{dy}{dx} + 2\frac{dy}{dx}$$

$$1 = \frac{dy}{dx}\left(3y^2 + 2\right)$$

$$\frac{1}{3y^2+2} = \frac{dy}{dx}$$

$$f'(x) = 3x^2 + 2$$

$$\frac{1}{3y^2 + 2}$$

27. Let f be the function defined by $f(x) = x^3 + x$. If $g(x) = f^{-1}(x)$ and $g(2) = 1$, what is the value of $g'(2)$?

(A) $\frac{1}{13}$ (B) $\frac{1}{4}$ (C) $\frac{7}{4}$ (D) 4 (E) 13

$$y = x^3 + x$$

$$x = y^3 + y$$

$$1 = 3y^2 \frac{dy}{dx} + 1 \frac{dy}{dx}$$

$$\frac{1}{3y^2+1} = \frac{dy}{dx} \quad \cancel{(3y^2+1)} = \frac{1}{3(1)^2+1} = \frac{1}{4}$$

$$\frac{d}{dx} f^{-1}(x) = \frac{1}{f'(f^{-1}(x))} = \frac{1}{f'(f^{-1}(2))} = \frac{1}{f'(1)} = \frac{1}{4}$$

EXAMPLE 5 **Derivative of an inverse function** The function $f(x) = \sqrt{x} + x^2 + 1$ is one-to-one for $x \geq 0$ and has an inverse on th interval. Find the slope of the curve $y = f^{-1}(x)$ at the point $\boxed{(3, 1).}$

$$x = \sqrt{y} + y^2 + 1$$

$$1 = \frac{1}{2} y^{-1/2} \frac{dy}{dx} + 2y \frac{dy}{dx}$$

$$1 = \frac{dy}{dx}\left(\frac{1}{2} y^{-1/2} + 2y\right)$$

$$\frac{1}{\frac{1}{2} y^{-1/2} + 2y} = \frac{dy}{dx} = \frac{1}{\frac{1}{2}(1)^{-1/2} + 2(1)} = \frac{1}{5/2} = 2/5$$

11. C If f and f^{-1} are both differentiable for all x, with $\boxed{f(3) = 5}$ and $\boxed{f'(3) = 7,}$ then which of the following must be a line tangent to the graph of f^{-1}?

(A) $y = 5 + 7(x - 3)$

(B) $y = \frac{1}{5} + \frac{1}{7}(x - 3)$

(C) $y = 3 + 7(x - 5)$

(D) $y = \frac{1}{3} + \frac{1}{7}(x - 5)$

(E) $y = 3 + \frac{1}{7}(x - 5)$

<u>Inverse</u>

$(5, 3)$

$m = \frac{1}{7}$

$y - 3 = \frac{1}{7}(x - 5)$

$g(1) = 2$

$g^{-1}(2) = 1$

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Question 3

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
1	6	4	2	5
2	9	2	3	1
3	10	–4	4	2
4	–1	3	6	7

$(2, 1)$

$m = 1/5$

The functions f and g are differentiable for all real numbers, and g is strictly increasing. The table above gives values of the functions and their first derivatives at selected values of x. The function h is given by $h(x) = f(g(x)) - 6$.

(a) Explain why there must be a value r for $1 < r < 3$ such that $h(r) = -5$.

(b) Explain why there must be a value c for $1 < c < 3$ such that $h'(c) = -5$.

$y - 1 = \frac{1}{5}(x - 2)$

(c) Let w be the function given by $w(x) = \int_1^{g(x)} f(t)\,dt$. Find the value of $w'(3)$.

(d) If g^{-1} is the inverse function of g, write an equation for the line tangent to the graph of $y = g^{-1}(x)$ at $x = 2$.

g^{-1}

Example

http://vimeo.com/18957161

Derivatives of Inverse Trigonometric Functions

$y = \sin^{-1}(x)$ is known as the inverse of

$y = \sin x$.

For this to be true, $x = \sin y$ is the same as $y = \sin^{-1}(x)$.

Lets try to develop a formula to find the derivative of arcsin.

$y = \sin x$

$$x = \sin y$$ switch x, y

$$1 = \cos y \frac{dy}{dx}$$ implicit deriv.

$$\frac{1}{\cos y} = \frac{dy}{dx}$$ divide by $\cos y$

$$\frac{1}{\sqrt{1-\sin^2 y}} = \frac{dy}{dx}$$ identity

$$\frac{1}{\sqrt{1-x^2}} = \frac{dy}{dx}$$ replace $\sin y = x$

Therefore we have:

$$\frac{\mathrm{d}}{\mathrm{dx}}\sin^{-1} u = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$$

Example 2

Find the derivative of $y = \sin^{-1}\left(3x^2\right)$.

$$y' = \frac{1}{\sqrt{1-(3x^2)^2}} \cdot 6x$$

$$= \frac{6x}{\sqrt{1-9x^4}}$$

These are the three main inverse trig derivatives studied in this course. They can all de developed the same way as arcsin!

$$\frac{d(\sin^{-1} u)}{dx} = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$$

$$\frac{d(\cos^{-1} u)}{dx} = \frac{-1}{\sqrt{1-u^2}}\frac{du}{dx}$$

$$\frac{d(\tan^{-1} u)}{dx} = \frac{1}{1+u^2}\frac{du}{dx}$$

$$y = \tan^{-1}(4x^3)$$

Find y'

$$y' = \frac{1}{1+(4x^3)^2} \cdot 12x^2$$

$$y' = \frac{12x^2}{1+16x^6}$$

$$\boxed{\frac{d}{dx} arc\csc x = \frac{d}{dx}\csc^{-1} x = \frac{-1}{|x|\sqrt{x^2-1}}}$$

$$\boxed{\frac{d}{dx} arc\sec x = \frac{d}{dx}\sec^{-1} x = \frac{1}{|x|\sqrt{x^2-1}}}$$

$$\boxed{\frac{d}{dx} arc\cot x = \frac{d}{dx}\cot^{-1} x = \frac{-1}{1+x^2}}$$

http://www.intmath.com/Differentiation-transcendental/3_Derivative-arcsin-arccos-arctan.php

$$= e^2\left(\frac{1}{2} + 2\left(\frac{\pi}{4}\right)\right)$$

$$= e^2\left(\frac{1}{2} + \frac{\pi}{2}\right)$$

1. If $f(x) = e^{2x}\tan^{-1}(x)$, then $f'(1) =$

(A) $\frac{e^2}{2}$ (B) $\frac{e^2\pi}{4}$ (C) e^2 (D) $\frac{e^2\pi}{2}$ (E) $\frac{e^2(\pi+1)}{2}$

$$f'(x) = e^{2x}\left(\frac{1}{1+x^2}\right) + \tan^{-1}(x)\, e^{2x}\cdot 2$$

$$= e^{2x}\left(\frac{1}{1+x^2} + 2\tan^{-1}(x)\right)$$

$$= e^{2(1)}\left(\frac{1}{1+1^2} + 2\tan^{-1}(1)\right)$$

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Question 3

A particle moves along the y-axis so that its velocity v at time $t \geq 0$ is given by $v(t) = 1 - \tan^{-1}\left(e^t\right)$. At time $\boxed{t = 0, \text{ the particle is at } y = -1.}$ (Note: $\tan^{-1} x = \arctan x$)

(a) Find the acceleration of the particle at time $t = 2$.

(b) Is the speed of the particle increasing or decreasing at time $t = 2$? Give a reason for your answer.

(c) Find the time $t \geq 0$ at which the particle reaches its highest point. Justify your answer.

(d) Find the position of the particle at time $t = 2$. Is the particle moving toward the origin or away from the origin at time $t = 2$? Justify your answer.

a) $a(t) = v'(t) = v'(2) = -.133$

b) $v(2) = -.436$

speed at 2 is inc because $a(2)$ and $v(2) < 0$

Assignment
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