3.7 Implicit Differentiation

So far we have described functions by expressing one variable explicitly in terms of another variable.

$$y = x^2 \text{ or } y = \frac{\sqrt{4 - x^2}}{x + 1}$$

Sometimes relations and/or functions are defined implicitly.

$$x^2 + y^2 = 25$$

We can still find the slope of a tangent line to this relation using implicit differentiation.

Ex.1 If $x^2 + y^2 = 25$ then :

a) Find $\frac{dy}{dx}$

b) Find the equation of the tangent line to the circle at the point $(-4,3)$.

a) $x^2 + y^2 = 25$

$2x + 2y \cdot \frac{dy}{dx} = 0$

$2y\frac{dy}{dx} = -2x$

$\frac{dy}{dx} = -\frac{2x}{2y} = -\frac{x}{y}$

$$\frac{dy}{dx} = \frac{-x}{y}$$

$$= \frac{-(-4)}{3} = \frac{4}{3}$$

$$y - y_1 = m(x - x_1)$$

$$y - 3 = \frac{4}{3}(x - (-4))$$

$$y - 3 = \frac{4}{3}(x + 4)$$

$$y - 3 = \frac{4}{3}x + \frac{16}{3}$$

$$\boxed{y = \frac{4}{3}x + \frac{25}{3}}$$

Implicit Differentiation Process

1. Differentiate both sides of the equation with respect to x.
2. Collect the terms with dy/dx on one side of the equation.
3. Factor out the dy/dx.
4. Solve for dy/dx.

Ex.2 Use implicit differentiation to find *dy/dx* for the following:

$$2x^5 + x^4 y + y^5 = 36$$

$$* \quad 10x^4 + x^4\left(1 \cdot \frac{dy}{dx}\right) + y(4x^3) + 5y^4 \frac{dy}{dx} = 0$$

$$10x^4 + x^4 \frac{dy}{dx} + 4x^3 y + 5y^4 \frac{dy}{dx} = 0$$

$$x^4 \frac{dy}{dx} + 5y^4 \frac{dy}{dx} = -10x^4 - 4x^3 y$$

$$\frac{dy}{dx}\left(x^4 + 5y^4\right) = -10x^4 - 4x^3 y$$

$$\boxed{\frac{dy}{dx} = \frac{-10x^4 - 4x^3y}{x^4 + 5y^4}}$$

Ex.3 Find *dy/dx* for the following:

$$x^3 - 4x^2 y^4 + 3y = 9$$

$$3x^2 - 4x^2\left(4y^3\frac{dy}{dx}\right) + y^4(-8x) + 3\frac{dy}{dx} = 0$$

$$3x^2 - 16x^2y^3\frac{dy}{dx} - 8xy^4 + 3\frac{dy}{dx} = 0$$

$$3\frac{dy}{dx} - 16x^2y^3\frac{dy}{dx} = 8xy^4 - 3x^2$$

$$\frac{dy}{dx}\left(3 - 16x^2y^3\right) = 8xy^4 - 3x^2$$

$$\frac{dy}{dx} = \frac{8xy^4 - 3x^2}{3 - 16x^2y^3}$$

AP Exam Free Response 2004 #4

4. Consider the curve given by $x^2 + 4y^2 = 7 + 3xy$.

(a) Show that $\frac{dy}{dx} = \frac{3y - 2x}{8y - 3x}$.

(b) Show that there is a point P with x-coordinate 3 at which the line tangent to the curve at P is horizontal. Find the y-coordinate of P.

(c) Find the value of $\frac{d^2y}{dx^2}$ at the point P found in part (b). Does the curve have a local maximum, a local minimum, or neither at the point P? Justify your answer.

a) $2x + 8y\frac{dy}{dx} = 0 + 3x \cdot \frac{dy}{dx} + y(3)$

$2x + 8y\frac{dy}{dx} = 3x\frac{dy}{dx} + 3y$

$8y\frac{dy}{dx} - 3x\frac{dy}{dx} = 3y - 2x$

$$\frac{dy}{dx}(8y-3x)=3y-2x$$

$$\frac{dy}{dx}=\frac{3y-2x}{8y-3x}$$

AP Exam MC 1997 #17

If $x^2 + y^2 = 25$, what is the value of $\frac{d^2y}{dx^2}$ at the point $(4,3)$

$$2x + 2y\frac{dy}{dx} = 0$$

$$\boxed{\frac{dy}{dx} = \frac{-x}{y}}$$

$$\frac{d^2y}{dx^2} = \frac{y(-1) - (-x)\left(\frac{dy}{dx}\right)}{(y)^2}$$

$$= \frac{-y + x\left(\frac{dy}{dx}\right)}{y^2}$$

$$= \frac{-y + x\left(\frac{-x}{y}\right)}{y^2} = \frac{-y - \frac{x^2}{y}}{y^2}$$

$$= \frac{-3 - \frac{(4)^2}{3}}{(3)^2} = \frac{-3 - \frac{16}{3}}{9}$$

$$= \frac{-\frac{9}{3} - \frac{16}{3}}{9}$$

$$= -\frac{25}{3} \cdot \frac{1}{9} = -\frac{25}{27}$$

Assignment
Calc 30 Text Page 211
#'s 8,9,11,14,16,17,20, 23, 27, 30, 32

AP Text Page 155
#'s 25, 29, 33, 36

Handout Questions

AP Exam MC 2003 #26

What is the slope of the line tangent to the curve $3y^2 - 2x^2 = 6 - 2xy$ at the point $(3,2)$?

a) 0 b) $\frac{4}{9}$ c) $\frac{7}{9}$ d) $\frac{6}{7}$ e) $\frac{5}{3}$

AP Question 1998 MC

6. If $x^2 + xy = 10$, then when $x = 2$, $\frac{dy}{dx} =$

(A) $-\frac{7}{2}$ (B) -2 (C) $\frac{2}{7}$ (D) $\frac{3}{2}$ (E) $\frac{7}{2}$

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Question 6

Consider the closed curve in the xy-plane given by

$$x^2 + 2x + y^4 + 4y = 5.$$

(a) Show that $\dfrac{dy}{dx} = \dfrac{-(x+1)}{2\left(y^3+1\right)}$.

(b) Write an equation for the line tangent to the curve at the point $(-2, 1)$.

(c) Find the coordinates of the two points on the curve where the line tangent to the curve is vertical.

(d) Is it possible for this curve to have a horizontal tangent at points where it intersects the x-axis? Explain your reasoning.

27. If $(x+2y)\cdot\frac{dy}{dx}=2x-y$, what is the value of $\frac{d^2y}{dx^2}$ at the point $(3,0)$?

(A) $-\frac{10}{3}$ (B) 0 (C) 2 (D) $\frac{10}{3}$ (E) Undefined

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Question 6

Solutions to the differential equation $\frac{dy}{dx} = xy^3$ also satisfy $\frac{d^2y}{dx^2} = y^3\left(1 + 3x^2y^2\right)$. Let $y = f(x)$ be a particular solution to the differential equation $\frac{dy}{dx} = xy^3$ with $f(1) = 2$.

(a) Write an equation for the line tangent to the graph of $y = f(x)$ at $x = 1$.

(b) Use the tangent line equation from part (a) to approximate $f(1.1)$. Given that $f(x) > 0$ for $1 < x < 1.1$, is the approximation for $f(1.1)$ greater than or less than $f(1.1)$? Explain your reasoning.

(c) Find the particular solution $y = f(x)$ with initial condition $f(1) = 2$.