3.6 The Chain Rule

The Chain Rule is used to take the derivative of composite functions

Say we want to find $\frac{dy}{dx}$ of the following function:

$$y = \left(2x^2 + 4\right)^8$$

let $u = 2x^2 + 4$

let $y = u^8$

$$\frac{du}{dx} = 4x$$

$$\frac{dy}{du} = 8u^7$$

$$\frac{dy}{du} \cdot \frac{du}{dx} = 8u^7 \cdot 4x$$

$$= 32x(2x^2+4)^7$$

$$y = (2x^2+4)^8$$

$$y' = 8(2x^2+4)^7 \cdot (4x)$$

$$= 32x(2x^2+4)^7$$

Example 1

Given that $y = u^{15}$ and $u = x^2 + 3$

find $\dfrac{dy}{dx}$.

Now to always differentiate a composite function this way would be time consuming. Therefore there must be an easier way!

http://www.mathtv.com/

The Chain Rule

If both $f(x)$ and $g(x)$ are differentiable functions, then if $F(x)=f(g(x))$, then $F'(x)=\underbrace{f'(g(x))}_{\text{outside}}\bullet\underbrace{g'(x)}_{\text{inside}}$.

Ex.2 Find the derivative of the following:

a) $y = \left(x^2 - x + 2\right)^8$

$y' = 8\left(x^2 - x + 2\right)^7 \cdot (2x - 1)$

b) $y = \dfrac{1}{\sqrt[3]{x^2 - 4x + 2}}$

No variable in numerator use Chain rule.

$$y = (x^2 - 4x + 2)^{-1/3}$$

$$y' = -\frac{1}{3}(x^2 - 4x + 2)^{-4/3}(2x - 4)$$

$$= \frac{-2(x-2)}{3(x^2 - 4x + 2)^{4/3}}$$

Example 6 A table of values for f, g, f', and g' is shown.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
3	5	6	8	4
4	1	3	5	2
5	3	9	7	10

a) If $F(x) = f(g(x))$, find $F'(4)$

$$F'(\cancel{x}4) = f'(g(\cancel{x}4)) \cdot g'(\cancel{x}4)$$
$$= f'(3) \cdot g'(4) = (8)(2) = 16$$

b) If $G(x) = g(f(x))$, find $G'(5)$

$$G'(\cancel{x}5) = g'(f(\cancel{x}5)) \cdot f'(\cancel{x}3)$$
$$= g'(3) \cdot f'(5)$$
$$= (4)(7) = 28$$

(c) Suppose the function g is defined by

$$g(x) = \begin{cases} k\sqrt{x+1} & \text{for } 0 \le x \le 3 \\ mx + 2 & \text{for } 3 < x \le 5, \end{cases}$$

where k and m are constants. If g is differentiable at $x = 3$, what are the values of k and m ?

$$\lim_{x \to 3^+} g(x) = \lim_{x \to 3^-} g(x)$$ b/c continuous

$$3m + 2 = k\sqrt{3+1}$$

$$\boxed{3m + 2 = 2k}$$

Derivative $k\sqrt{x+1}$ = derv. $mx+2$

at $x = 3$.

$$K(x+1)^{1/2} = mx + 2$$

$$\frac{1}{2}K(x+1)^{-1/2} = m$$

$$\frac{K}{2(x+1)^{1/2}} = m$$

$$\frac{K}{2(3+1)^{1/2}} = m$$

$$\frac{K}{4} = m$$

$$3\left(\frac{K}{4}\right) + 2 = 2K$$

$$\frac{3K}{4} + 2 = 2K$$

$$2 = \frac{5K}{4}$$

$$8 = 5K$$

$$8/5 = K$$

$$\frac{\left(\frac{8}{5}\right)}{4} = m$$

$$\frac{8}{20} = m$$

$$2/5 = m$$

Assignment Page 200 Calc 30 Text

#'s

1,4,6,11,13,15,18,20,24,25,26,28

$$y = (3x^2 + 5x)^4$$

$$y' = 4(3x^2 + 5x)^3 \cdot (6x + 5)$$

Use the chain rule to find the derivative of each of the following functions. You may leave negative and rational exponents in your answer.

1. $y=(3x-4)^6$
2. $y=(5-2x)^{11}$
3. $y=(x+7)^6$
4. $y=(2-x)^5$
5. $f(x)=(x^2+3)^3$
6. $f(x)=(3x^2-5)^4$
7. $y=(12-2x^3)^5$
8. $f(x)=(x^2-2x-3)^4$
9. $f(x)=(x^4-8x)^7$
10. $y=(3x-x^2)^3$
11. $y=(x^2+4)^{-3}$
12. $y=(x^2+6x-3)^{-2}$
13. $f(x)=\dfrac{3}{x^2+9}$
14. $f(x)=\dfrac{1}{5x^2+10x+4}$
15. $f(x)=\sqrt{x^2+6x}$
16. $f(x)=-3\sqrt{2x^2-4x-1}$
17. $y=\dfrac{6}{5}\sqrt[3]{x-2}$
18. $y=9\sqrt[3]{(x^2-4)^2}$
19. $y=\dfrac{10}{\sqrt{2x-1}}$
20. $y=\dfrac{-8}{\sqrt[4]{2x^2-8x+1}}$
21. $f(x)=\left(x+\dfrac{1}{x}\right)^{1/2}$
22. $f(x)=12(x^3-6x)^6$
23. $f(x)=6\sqrt{\frac{1}{2}x-4}$
24. $f(x)=\dfrac{5}{3}\sqrt[5]{(5x^2+10x-1)^3}$

25. Find the slope of the tangent line to the curve $f(x)=2\sqrt{x^2+13}$ drawn at the point $(6, f(6))$.

26. Find the coordinates of the point(s) on the curve $f(x)=\sqrt[3]{x^2-4x+3}$ at which the tangent line is:

(a) horizontal.
(b) vertical.

27. Find the derivative of the function $f(x)=\dfrac{5}{3x^2+6x}$ using the:

(a) quotient rule.
(b) chain rule.

28. The table at right shows the values of $f(x)$, $g(x)$, $f'(x)$, and $g'(x)$ for various values of x. Find:

(a) $(f\circ g)'(2)$.

(b) $(g\circ f)'(3)$

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
1	4	8	6	−4
2	3	1	3	−3
3	1	7	5	−5

Sometimes we have a combination of the chain rule with another derivative rule.

Ex.3 Find the derivative of the following:

$$s=\left(\frac{2t-1}{t+2}\right)^6$$

$$s'=6\left(\frac{2t-1}{t+2}\right)^5\left[\frac{(t+2)(2)-(2t-1)(1)}{(t+2)^2}\right]$$

$$=\frac{6(2t-1)^5}{(t+2)^5}\left[\frac{2t+4-2t+1}{(t+2)^2}\right]$$

$$=\frac{6(2t-1)^5}{(t+2)^5}\left(\frac{5}{(t+2)^2}\right)=\frac{30(2t-1)^5}{(t+2)^7}$$

Ex.4 Find the derivative of the following:

$$f(x)=\underbrace{\left(x^2+1\right)^3}\underbrace{\left(2-3x\right)^4}$$

$$f'=(x^2+1)^3\cdot 4(2-3x)^3(-3)+(2-3x)^4\cdot 3(x^2+1)^2\cdot 2x$$

$$f'=-12(x^2+1)^3(2-3x)^3+6x(x^2+1)^2(2-3x)^4$$

GCF

$$=6(x^2+1)^2(2-3x)^3\left[-2(x^2+1)+x(2-3x)\right]$$

$$=6(x^2+1)^2(2-3x)^3(-2x^2-2+2x-3x^2)$$

$$=6(x^2+1)^2(2-3x)^3(-5x^2+2x-2)$$

2. If $f(x) = x\sqrt{2x - 3}$, then $f'(x) =$

(A) $\dfrac{3x - 3}{\sqrt{2x - 3}}$

(B) $\dfrac{x}{\sqrt{2x - 3}}$

(C) $\dfrac{1}{\sqrt{2x - 3}}$

(D) $\dfrac{-x + 3}{\sqrt{2x - 3}}$

(E) $\dfrac{5x - 6}{2\sqrt{2x - 3}}$

$$f(x) = x(2x-3)^{1/2}$$

$$f'(x) = x \cdot \frac{1(\cancel{2})}{\cancel{2}(2x-3)^{1/2}} + (2x-3)^{1/2}(1)$$

$$= \frac{x}{(2x-3)^{1/2}} + \frac{(2x-3)^{1/2}(2x-3)^{1/2}}{(2x-3)^{1/2}}$$

$$= \frac{x + (2x-3)}{(2x-3)^{1/2}} = \frac{3x-3}{\sqrt{2x-3}}$$

Ex.5 Find the derivative of the following:

$$y=\frac{\sqrt{x^2+3x}}{(x^2+2)^2} = \frac{(x^2+3x)^{1/2}}{(x^2+2)^2}$$

$$\cancel{2(x^2+3x)^{1/2}} \qquad\qquad 2(x^2+3x)^{1/2}$$

$$y' = \frac{(x^2+2)^{\cancel{2}} \cdot \dfrac{1(2x+3)}{\cancel{2(x^2+3x)^{1/2}}} - (x^2+3x)^{1/2} \cdot 2(x^2\cancel{+2}) \cdot 2x}{2(x^2+3x)^{1/2}(x^2+2)^{3}}$$

$$= \frac{(x^2+2)(2x+3) - 8x(x^2+3x)}{2(x^2+3x)^{1/2}(x^2+2)^{3}}$$

$$= \frac{2x^3 + 3x^2 + 4x + 6 - 8x^3 - 24x^2}{2(x^2+3x)^{1/2}(x^2+2)^3}$$

$$= \frac{-6x^3 - 21x^2 + 4x + 6}{2(x^2+3x)^{1/2}(x^2+2)^3}$$

Assignment

Handout #'s 6 a, c, d, e, f, g, j, 7, 8, 9

Calc 30 Text Page 205 #'s 5, 10, 13, 15, 18