3.4 Velocity and Other Rates of Change

$$\text{ave} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

Recall from 2.4 the difference between average rate change and instantaneous rate change.

derivative

Ex. 1 Find the rate change of the area of a circle with respect to its radius. Evaluate the rate change of A at r=5m and r=10m.

$$A = \pi r^2$$

$$\frac{dA}{dr} = 2\pi r$$

$$\frac{dA}{dr} = 2\pi(5) = 10\pi \frac{m^2}{m}$$

$$\frac{dA}{dr} = 2\pi(10) = 20\pi \frac{m^2}{m}$$

displacement

In calculus we usually define the position of a function using the notation *s(t)*.

Instantaneous Velocity is the derivative of the position function *s(t)* with respect to time.

$$v(t) = \frac{ds}{dt} = s'(t)$$

Speed is defined as the absolute value of velocity

$$\text{speed} = |v(t)| = \left|\frac{ds}{dt}\right| = |s'(t)|$$

Acceleration is defined as the derivative of velocity with respect to time.

$$a(t) = \frac{dv}{dt} = v'(t) = \frac{d^2s}{dt^2} = s''(t)$$

We usually deal with two types of position functions in calculus, **horizontal** and **vertical.**

Horizontal motion we usually denote by ***s(t)***.

Vertical motion we usually denote by ***h(t)***.

Ex. 1 A ball rolls along the x-axis so that its position is centimetres after t seconds is given by the function: $s(t) = t^3 + 3t^2$, where $t \geq 0$.

a) Find an expression for the velocity at any time t.

$$v(t) = 3t^2 + 6t$$

b) Find the velocity when $t=5$.

$$v(5) = 3(5)^2 + 6(5)$$
$$= 105 \text{ cm/s}$$

c) Find the average velocity between $t=1$ and $t=5$.

$$\text{ave vel} = \frac{s(t_2)-s(t_1)}{t_2-t_1} = \frac{s(5)-s(1)}{4}$$

$$= \frac{\left[(5)^3+3(5)^2\right]-\left[(1)^3+3(1)^2\right]}{4} = \frac{200-4}{4}$$

$$= 49 \text{ cm/s}$$

d) Where is the ball when its velocity is 72 cm/s?

$s(t) = ?$

$$72 = 3t^2+6t$$

$$0 = 3t^2+6t-72$$

$$0 = 3(t^2+2t-24)$$

$$0 = 3(t+6)(t-4)$$

$t=-6$ or $t=4$

$$S(4) = (4)^3 + 3(4)^2$$

$$= 64 + 48$$

$$= 112 \text{ cm}$$

e) Find an expression for the acceleration at any time t.

$$a(t) = v'(t) = 6t + 6$$

f) What is the acceleration of the ball at $t=1$?

$$a(1) = 6(1) + 6 = 12 \text{ cm/s}^2$$

g) What is the velocity of the ball when its acceleration is 54 cm/s^2?

$v(t) = ?$

$54 = 6t + 6$

$48 = 6t$

$8 = t$

$v(8) = 3(8)^2 + 6(8)$

$= 192 + 48 = 240$ cm/s

h) Find the average acceleration between $t=1$ and $t=5$.

$$\text{ave accel} = \frac{v(t_2) - v(t_1)}{t_2 - t_1} = \frac{v(5) - v(1)}{5 - 1}$$

$$= \frac{[3(5)^2 + 6(5)] - [3(1)^2 + 6(1)]}{4} = \frac{105 - 9}{4} = 24 \text{ cm/s}^2$$

Ex.2 A rock is thrown upwards from the top of a house and its height above ground in metres after t seconds is given by the function: $h(t) = -5t^2 + 40t + 5$

a) Find the initial height of the rock?

Set $t = 0$ $\quad h(0) = -5(0) + 40(0) + 5$

$h(0) = 5m$

b) Find the velocity of the rock after 2 seconds.

$V(t) = h'(t) = -10t + 40$

$V(2) = -10(2) + 40 = 20m/$

c) When does the rock reach its maximum height?

Set $v(t) = 0$

$$-10t + 40 = 0$$

$$40 = 10t$$

$$4s = t$$

d) What is the maximum height of the rock?

$$h(4) = -5(4)^2 + 40(4) + 5$$

$$= -80 + 160 + 5 = 85\,m$$

e) When does the rock hit the ground?

Set $h(t) = 0$

$$-5t^2 + 40t + 5 = 0$$

$t = -.123$ s ~~or~~ or $t = 8.123$ s

f) With what velocity does the rock hit the ground?

$$V(8.123) = -10(8.123) + 40$$

$$= -41.23 \text{ m/s}$$

Your Turn #4

Suppose that a bee flies along the x-axis so that its position at time, t, is given by the function $s(t) = 2t^3 - 21t^2 + 60t$.

(a) Where is the bee when $t = 3$?

(b) What is the velocity of the bee when $t = 3$?

(c) What is the velocity of the bee when $t = 6$?

(d) Find the time interval(s) when the bee is moving to the right.

(e) Find the time interval(s) when the bee is moving to the left.

(f) Show that the bee has moved a total of 90 units during the time interval $t = 0$ to $t = 6$.

a) $S(3) = 2(3)^3 - 21(3)^2 + 60(3)$

$= 45$ cm

b) $V(t) = 6t^2 - 42t + 60$

$V(3) = -12$ cm/s

c) $V(6) = 24$ cm/s

d) Set $v(t) = 0$

e) $6t^2 - 42t + 60 = 0$

$6(t^2 - 7t + 10) = 0$

$6(t-5)(t-2) = 0$

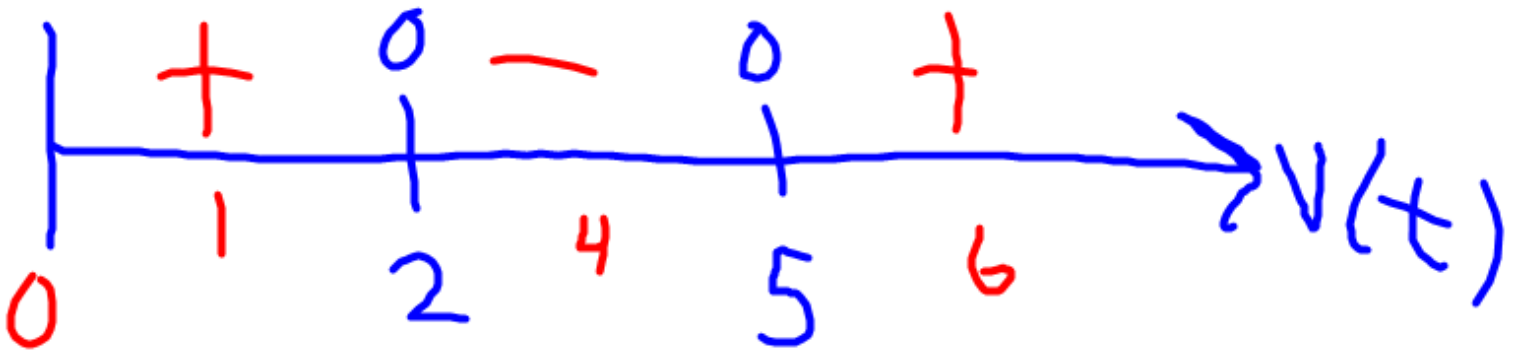
$t = 2$ or $t = 5$

Right

$(0,2) \cup (5,\infty)$

Left

$(2,5)$



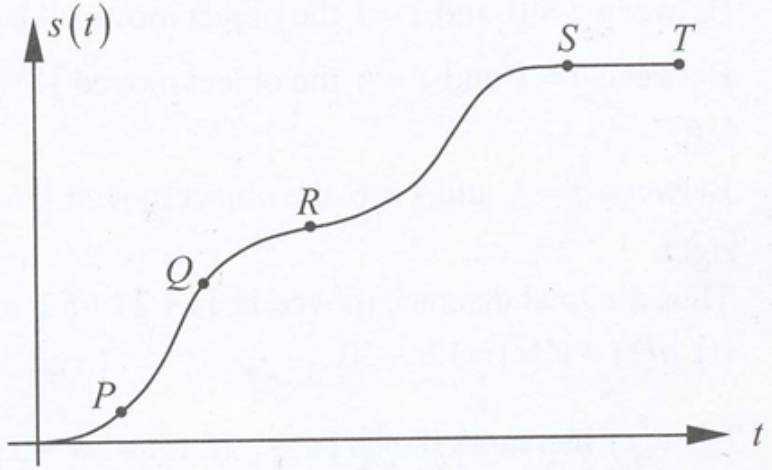
$$|s(2)-s(0)| = 52\,cm$$

$$|s(5)-s(2)| = |-27| = 27\,cm$$

$$|s(6)-s(5)| = 11\,cm$$

$$90\,cm$$

1. The graph at right shows a car's distance from town as it travelled along a straight stretch of highway. Answer each of the following questions explaining your conclusions.
   (a) What was the initial velocity of the car?
   (b) Was the car going faster at Q or at R?
   (c) Was the car going faster at P or at Q?
   (d) Was the car speeding up or slowing down at P?
   (e) Was the car speeding up or slowing down at Q?
   (f) Was the car speeding up or slowing down at R?
   (g) What happened to the car between S and T?



2003 MC No Calculator

25. A particle moves along the x-axis so that at time $t \geq 0$ its position is given by $x(t) = 2t^3 - 21t^2 + 72t - 53$. At what time t is the particle at rest?

(A) $t = 1$ only

(B) $t = 3$ only

(C) $t = \frac{7}{2}$ only

(D) $t = 3$ and $t = \frac{7}{2}$

(E) $t = 3$ and $t = 4$


I TOLD YOU CIGARS WERE BAD FOR YOU!

2003 MC Calculator Allowed

76. A particle moves along the x-axis so that at any time $t \geq 0$, its velocity is given by $v(t) = 3 + 4.1\cos(0.9t)$. What is the acceleration of the particle at time $t = 4$?

(A) -2.016 (B) -0.677 (C) 1.633 (D) 1.814 (E) 2.978

16. A particle moves along the x-axis with its position at time t given by $x(t) = (t - a)(t - b)$, where a and b are constants and $a \neq b$. For which of the following values of t is the particle at rest?

(A) $t = ab$

(B) $t = \frac{a + b}{2}$

(C) $t = a + b$

(D) $t = 2(a + b)$

(E) $t = a$ and $t = b$

$$x(t) = t^2 - at - bt + ab$$

$$v(t) = x'(t) = 2t - a - b$$

$$2t - a - b = 0$$

$$2t = a + b$$

$$t = \frac{a + b}{2}$$

28. For $t \geq 0$, the position of a particle moving along the x-axis is given by $x(t) = \sin t - \cos t$. What is the acceleration of the particle at the point where the velocity is first equal to 0 ?

(A) $-\sqrt{2}$ (B) -1 (C) 0 (D) 1 (E) $\sqrt{2}$

$x(t) = \sin t - \cos t$

$v(t) = \cos t + \sin t$

$a(t) = -\sin t + \cos t$

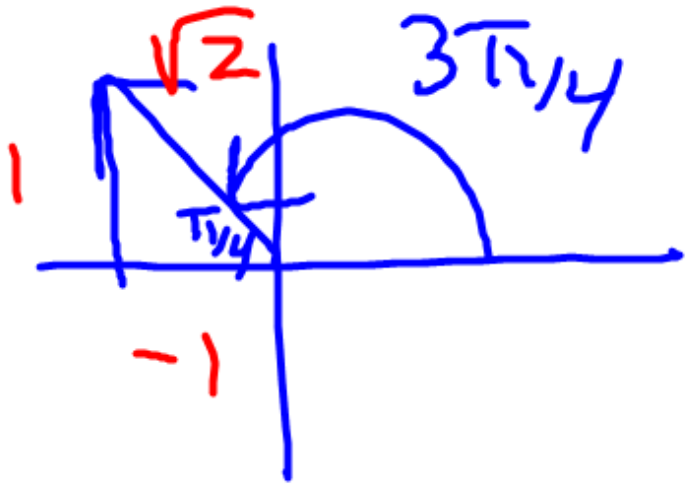
$\cos t + \sin t = 0$

$\cos t = -\sin t$

$t = 3\pi/4$

unit circle

$$a(3\pi/4) = -\left(\sin\frac{3\pi}{4}\right) + \left(\cos 3\pi/4\right)$$



$$= -\frac{1}{\sqrt{2}} + -\frac{1}{\sqrt{2}}$$

$$= -\frac{2}{\sqrt{2}} = -\frac{2\sqrt{2}}{2} = -\sqrt{2}$$

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2. For $t \geq 0$, a particle moves along the x-axis. The velocity of the particle at time t is given by $v(t) = 1 + 2\sin\left(\frac{t^2}{2}\right)$. The particle is at position $x = 2$ at time $t = 4$.

(a) At time $t = 4$, is the particle speeding up or slowing down?

(b) Find all times t in the interval $0 < t < 3$ when the particle changes direction. Justify your answer.

(c) Find the position of the particle at time $t = 0$.

(d) Find the total distance the particle travels from time $t = 0$ to time $t = 3$.

a)

$v(4) = 2.979 > 0$

$a(4) = -1.164 < 0$

Since the $v(4)$ and $a(4)$ have different signs particle slowing down.

b) $t = 2.71$

that is where $v(t)$ changes from positive to negative

b)

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Question 4

A particle moves along the x-axis with position at time t given by $x(t) = e^{-t}\sin t$ for $0 \le t \le 2\pi$.

(a) Find the time t at which the particle is farthest to the left. Justify your answer.

(b) Find the value of the constant A for which $x(t)$ satisfies the equation $Ax''(t) + x'(t) + x(t) = 0$ for $0 < t < 2\pi$.

a) $v(t) = e^{-t}(\cos t) + \sin t\,(-e^{-t})$

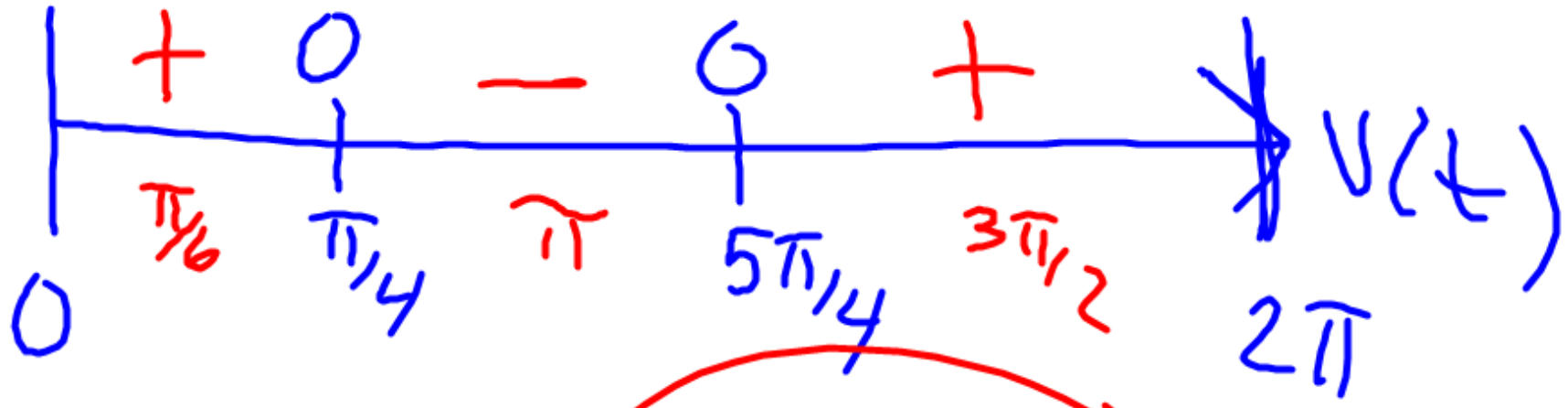
$v(t) = e^{-t}(\cos t - \sin t)$

new $= 0$.

$$\cos t - \sin t = 0$$

$$\cos t = \sin t$$

$t = \pi/4$ $\quad$ $t = 5\pi/4$



$\cos \pi/6 - \sin \pi/6$

$\sqrt{3}/2 - 1/2$

$t = 5\pi/4$

$$A\,x''(t) + x'(t) + x(t) = 0$$

$$x'(t) = e^{-t}(\cos t - \sin t)$$

$$x''(t) = e^{-t}(-\sin t - \cos t) + (\cos t - \sin t)(-e^{-t})$$

$$= e^{-t}(-\sin t - \cos t - \cos t + \sin t)$$

$$= e^{-t}(-2\cos t)$$

$$x''(t) = -2e^{-t}\cos t$$

$$A(-2e^{-t}\cos t) + e^{-t}(\cos t - \sin t) + e^{-t}\sin t = 0$$

$$e^{-t}(-2A\cos t + \cos t - \sin t + \sin t) = 0$$

$e^{-t} \neq 0$

$$(-2A\cos t + \cos t) = 0$$

$$-2A\cos t + \cos t = 0$$

$$\cos t(-2A + 1) = 0$$

$$-2A + 1 = 0$$

$$A = 1/2$$

Assignment

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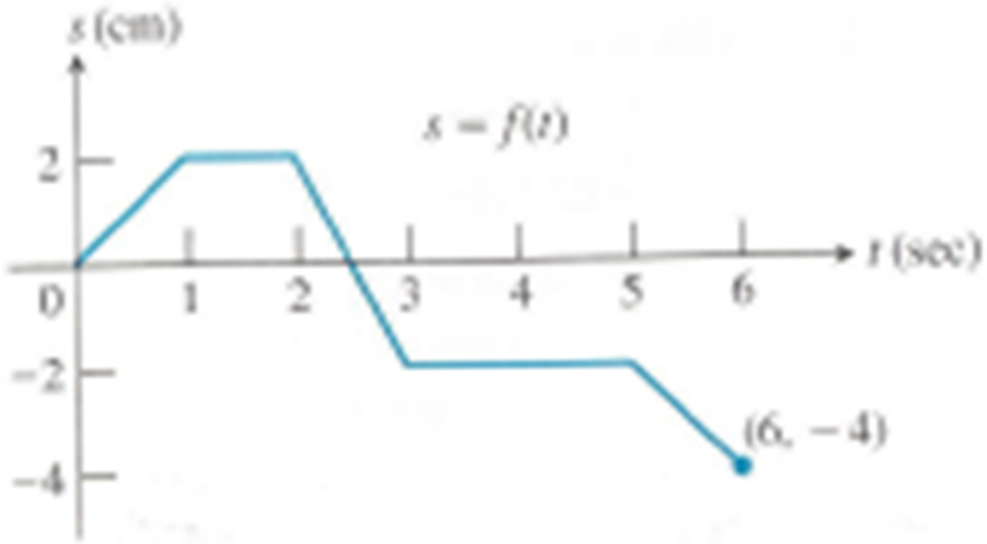
AP

#'s

2omit f,3,4,5,6,8,9,12,13,24,25,31a

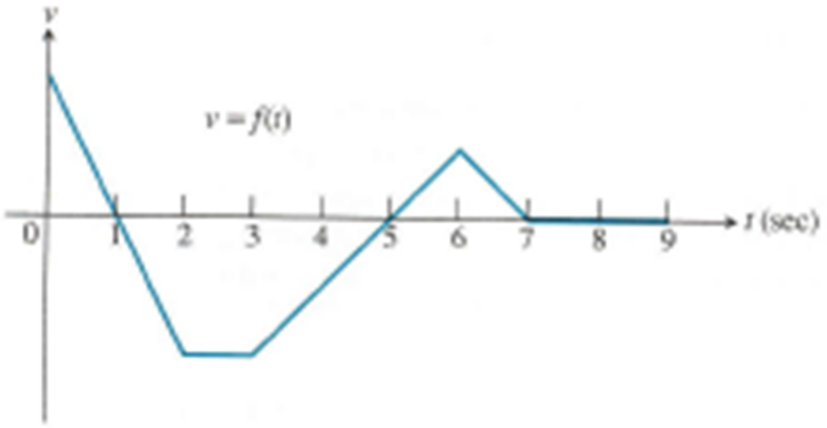


(b)

(a) When is P moving to the left? moving to the right? standing still?

(b) Graph the particle's velocity and speed (where defined).

25. *Particle Motion* The accompanying figure shows the velocity $v = f(t)$ of a particle moving on a coordinate line.



(a) When does the particle move forward? move backward? speed up? slow down?

(b) When is the particle's acceleration positive? negative? zero?

(c) When does the particle move at its greatest speed?

(d) When does the particle stand still for more than an instant?