3.4 Continuity of Functions

3.4 Continuity

Learning Targets:

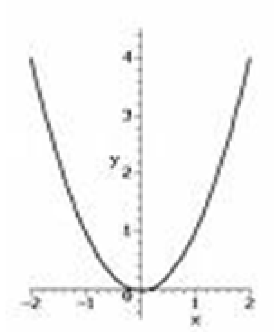
1. SWBAT define a continuous funciton.

2. SWBAT determine the location(s) and type(s) of discontinuities different functions may have.

http://www.calculus-help.com/continuity/

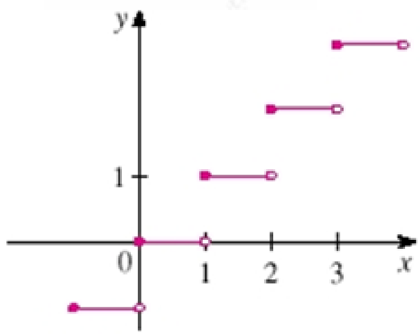
Defn: **Continuous functions** are ones whose graphs can be drawn without lifting your pencil off the paper.

$$\text{Example } y = x^2$$



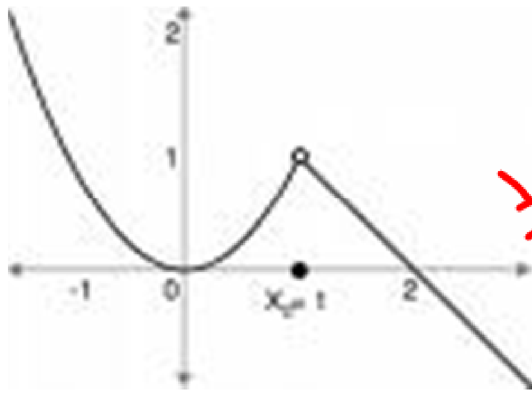
$$\text{Non - example } y = [x]$$

Greatest Integer Function



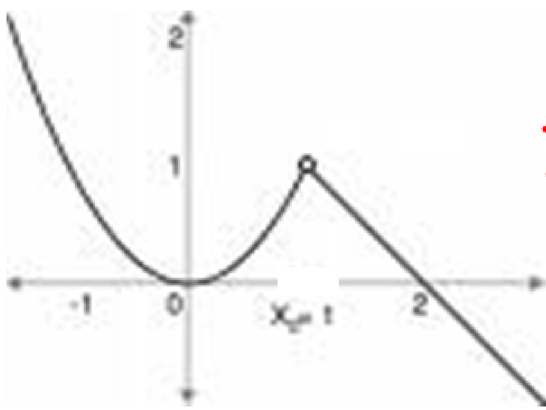
(d) $f(x) = [\![x]\!]$

Types of Discontinuities



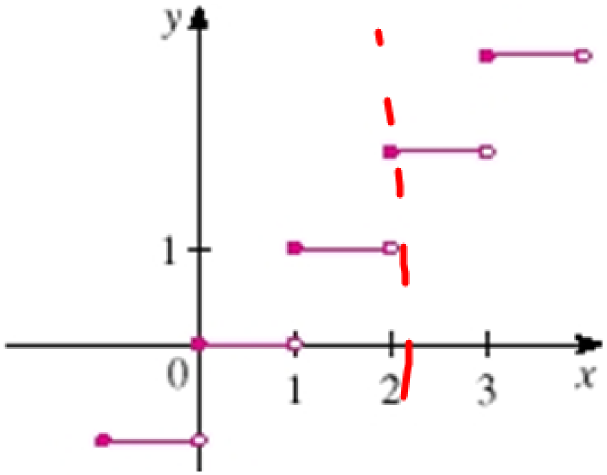
Removable Discontinuity

Limit exists at x=1 but the value of the function does not equal the limit at x=1.



Removable Discontinuity

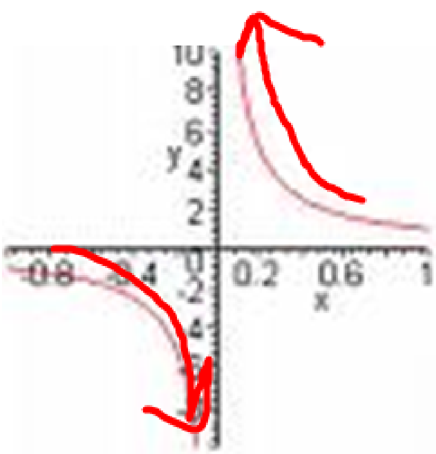
Limit exists x=1 but the value of the function does not exist at x=1.



(d) $f(x) = [\![x]\!]$

Jump Discontinuity

Limit from the left does not equal the limit from the right.



Infinite Discontinuity

Function is not defined at a vertical asymptote.

Example 1:

Show that the function $h(x) = \dfrac{x^3 - 125}{x - 5}$ has a removable discontinuity at $x = 5$. How should $h(5)$ be defined so that the discontinuity can be removed? What are the coordinates of the hole?

$$\lim_{x \to 5} \frac{x^3 - 125}{x - 5} = \lim_{x \to 5} \frac{\cancel{(x-5)}(x^2 + 5x + 25)}{\cancel{(x-5)}}$$

$$= \lim_{x \to 5} (x^2 + 5x + 25)$$

$$= (5)^2 + 5(2) + 25$$

$$= 75$$

(5, 75) hole

Example 2:

Find the location of the infinite discontinuities for the function $f(x) = \dfrac{3x+1}{x^2+x-30}$.

$$= \frac{3x+1}{(x+6)(x-5)}$$

$$\left.\begin{aligned} x &= -6 \\ x &= 5 \end{aligned}\right\} \text{infinite discont.}$$

Abs value $\rightarrow$ jump dis.

Example 3

Show that the function

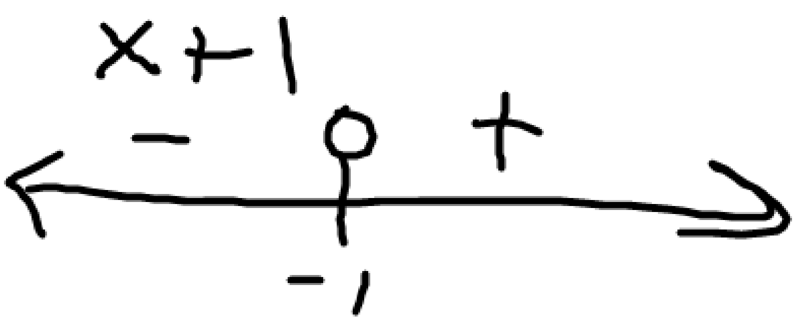
$$g(x) = \frac{x^2+3x+2}{|x+1|}$$ has a

jump discontinuity at $x=-1$ by comparing $\lim_{x\to -1^+} g(x)$ with $\lim_{x\to -1^-} g(x)$.

$$\lim_{x\to -1^+} \frac{x^2+3x+2}{|x+1|} = \lim_{x\to -1^+} \frac{\cancel{(x+1)}(x+2)}{\cancel{(x+1)}}$$

$$= -1+2$$

$$= 1$$



$$\lim_{x \to -1^-} \frac{x^2+3x+2}{|x+1|}$$

$$\lim_{x \to -1^-} \frac{(x+2)\cancel{(x+1)}}{-\cancel{(x+1)}} = \frac{(-1+2)}{-1}$$

$$= -1$$

Jump at $x = -1$

$$\lim_{x \to -1^+} f(x) \neq \lim_{x \to -1^-} f(x)$$

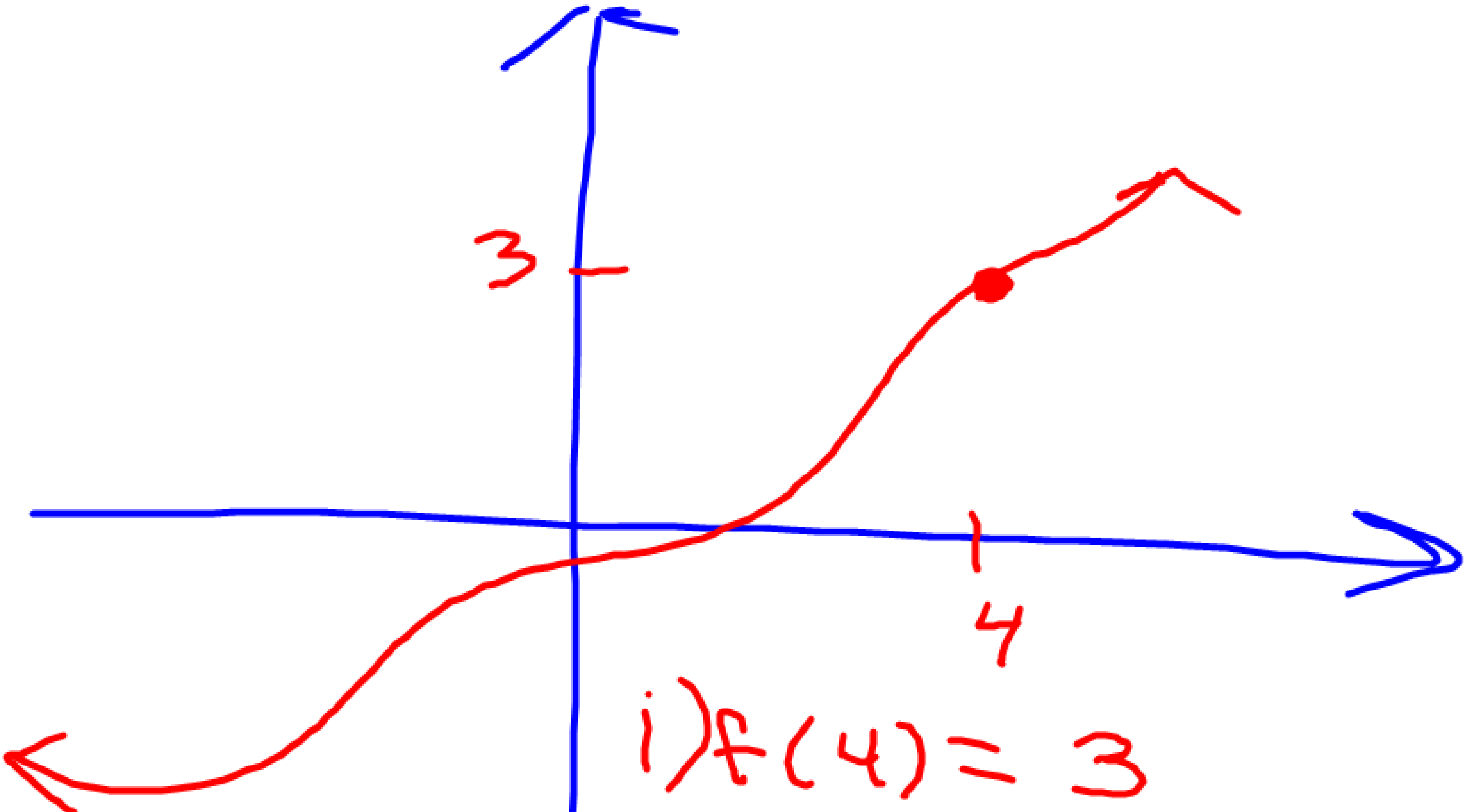
Mathematically, a function is continuous at a number "*a*" if: (a point)

i) $f(a)$ is defined

ii) $\lim\limits_{x \to a} f(x)$ exists

iii) $\lim\limits_{x \to a} f(x) = f(a)$

This is also known as the Test for Continuity at a Point.

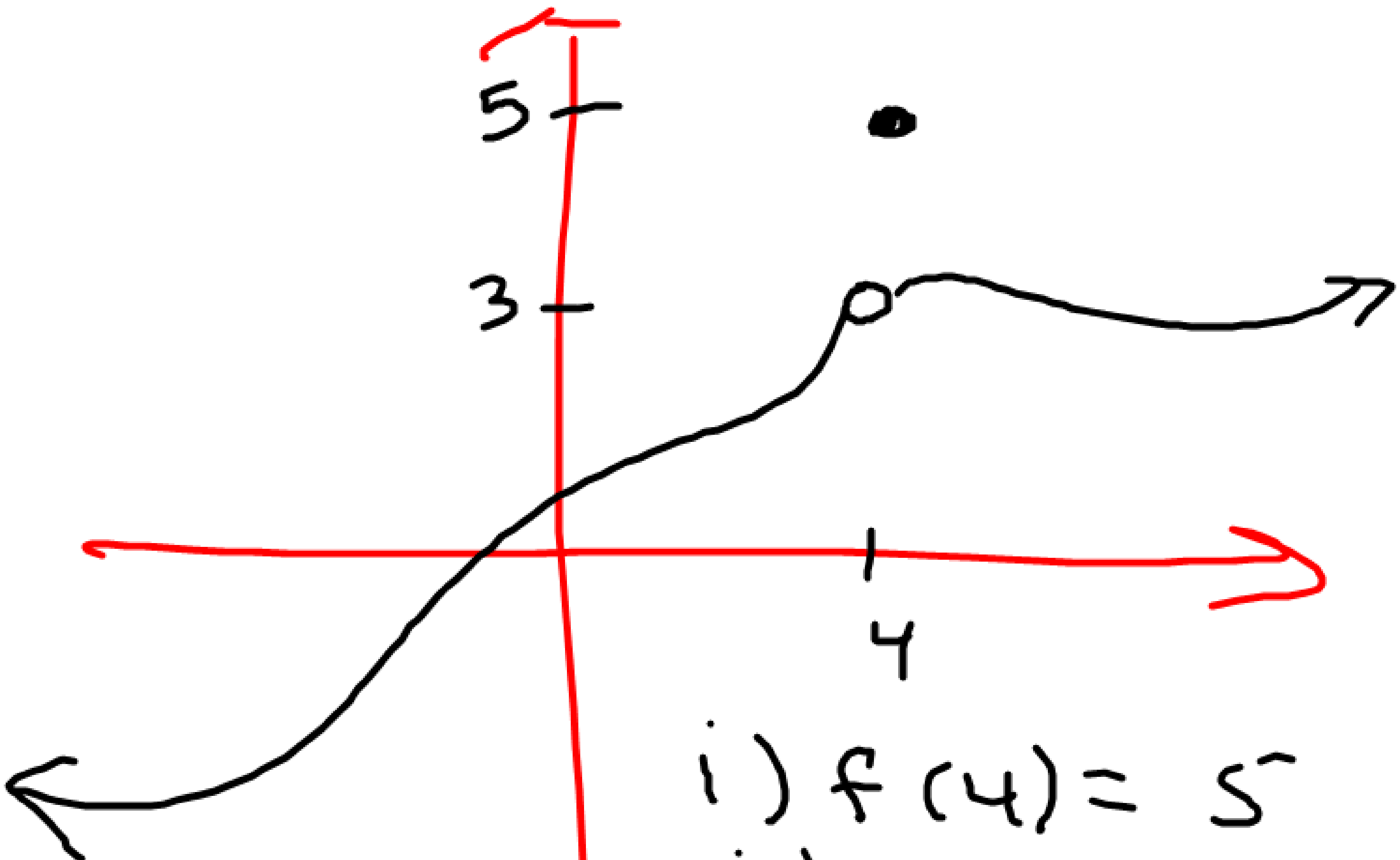


cont $x=4$

i) $f(4)=3$

ii) $\lim\limits_{x\to 4} f(x)=3$

iii) $f(4)=\lim\limits_{x\to 4} f(x)$



i) $f(4) = 5$

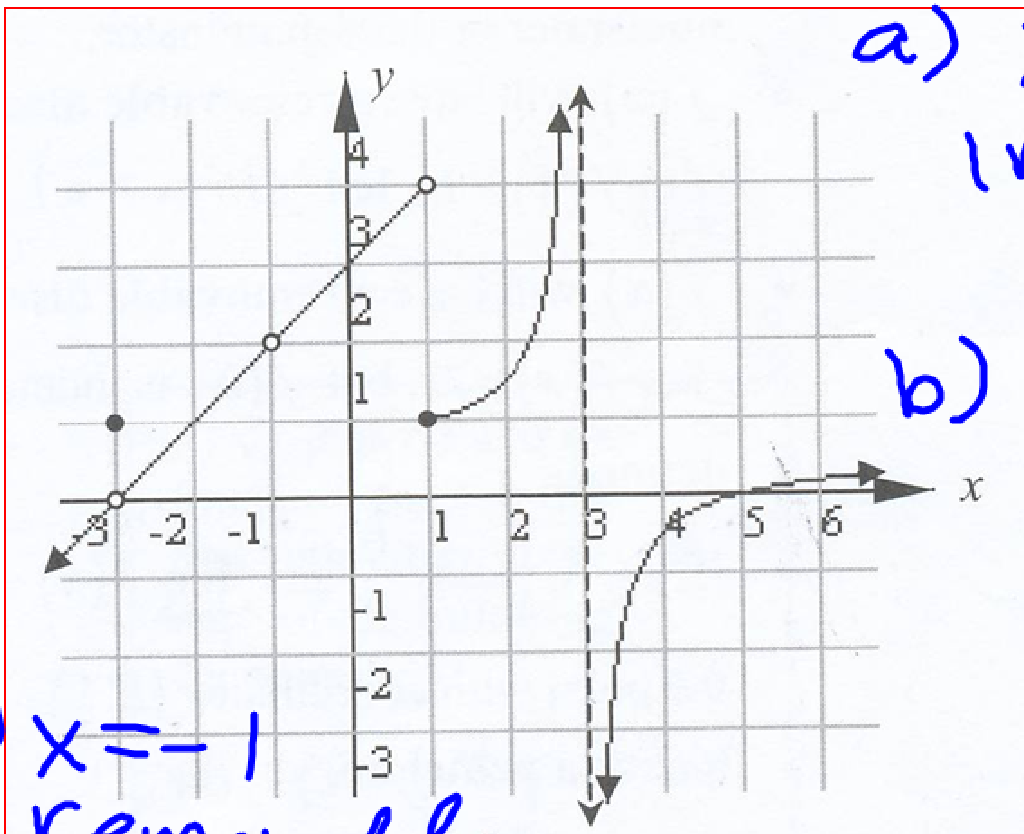
ii) $\lim_{x \to 4} f(x) = 3$

A function is said to be continuous if it is continuous at every point in its domain.

Continuity Principles

1. All constant functions are continuous.
2. The following types of functions are continuous at every point in their domain: polynomial, rational, power, root, trigonometric, exponential, and logarithmic.
3. If $f(x)$ and $g(x)$ are continuous functions, so are $(f+g)(x)$, $(f-g)(x)$, $(fg)(x)$, and $\left(\frac{f}{g}\right)(x)$ in their common domains provided, in the last case, that $g(x) \neq 0$.

Ex.1 By examining the following graph of *m(x)* explain why *m(x)* is discontinuous at the following points: a) x = 3 b) x = 1 c) x = -1 d) x = - 3. Classify each discontinuity.



a) $x = 3$
infinite
i) $\cancel{f(3)} =$

b) $x = 1$ jump
i) $m(1) = 1$
ii) $\cancel{\lim_{x \to 1}} m(x)$

c) $x = -1$
removable
i) $\cancel{f(-1)}$

d) $x = -3$

remvable

i) $m(-3) = 1$

ii) $\lim_{x \to -3} m(x) = 0$

iii) $\cancel{m(-3) = \lim_{x \to -3} m(x)}$

Tips For Finding Points Of Discontinuity

- Many functions are continuous. Become very familiar with your continuity principles so that you don't waste time looking for points of discontinuity that don't exist.
- Factoring the numerator and the denominator is helpful.
- $f(x)$ will have an **infinite discontinuity** at $x=b$ if $f(b)=\dfrac{\text{something } \neq 0}{0}$. If the denominator of a function can never be 0, it will not have an infinite discontinuity.
- $f(x)$ will have a **jump discontinuity** at $x=b$ if $\lim\limits_{x\to b^+} f(x)$ exists, $\lim\limits_{x\to b^-} f(x)$ exists, but $\lim\limits_{x\to b^+} f(x) \neq \lim\limits_{x\to b^-} f(x)$. Jump discontinuities occur most frequently with piecewise functions and functions containing absolute value signs in either the numerator or the denominator.
- $f(x)$ will have a **removable discontinuity** at $x=b$ if $\lim\limits_{x\to b} f(x)=L$, but $f(b)=T\neq L$.
- $f(x)$ will have a **removable discontinuity** at $x=b$ if $\lim\limits_{x\to b} f(x)=L$, but $f(b)$ is indeterminate or has not been defined.
- If $f(b)$ yields $\dfrac{0}{0}$ and $\lim\limits_{x\to b} f(x)=L$, then $f(x)$ has a hole at the point with coordinates (b,L). We will call this the **hole location principle**.

Ex.2 Which condition of the test for continuity at a point is not satisfied by each function at x = -1?

removable at $x = -1$

$a)\ f(x) = \frac{x-1}{x+1}$

$\cancel{f(-1)} = \frac{-1-1}{-1+1}$

$= \frac{-2}{0}$

$x = -1$ VA

infinite disc.

$b)\ g(x) = \begin{cases} \frac{x^2+5x+4}{x+1}, & x \neq -1 \\ 2, & x = -1 \end{cases}$

i) $g(-1) = 2$

ii) $\lim_{x \to -1} \frac{x^2+5x+4}{x+1}$

$\lim_{x \to -1} \frac{(x+4)\cancel{(x+1)}}{\cancel{(x+1)}}$

$= 3$

$\cancel{\text{iii)}}$ $g(-1) \neq \lim_{x \to -1} g(x)$

Ex.3 Find the value of "a" so that our function is continuous.

$$a)\ f(x)=\begin{cases} x^2, x<1 \\ \cancel{2ax-1}, x\geq 1 \end{cases}$$

$2x-1$

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x)$$

$$\lim_{x\to 1^-} x^2 = \lim_{x\to 1^+} 2ax-1$$

$$\boxed{1^2 = 2a(1)-1}$$

$$1 = 2a-1 \rightarrow 2=2a$$

$$1=a$$

$$b)\ f(x)=\begin{cases} -x^2+2, & x<2 \\ \dfrac{1}{2}ax-3, & x\geq 2 \end{cases}$$

$$\lim_{x\to 2^-} -x^2+2 = \lim_{x\to 2^+} \frac{1}{2}ax-3$$

$$-(2)^2+2 = \frac{1}{2}(2)a-3$$

$$-2 = a-3$$

$$1=a$$

$$c)\ f(x) = \left\{ \begin{array}{l} x^2 - 3,\ x \le -1 \\ cx - d,\ -1 < x < 3 \\ x^2 + 1,\ x \ge 3 \end{array} \right\}$$

Ex.4 Find and classify the discontinuities, if any, for each of the following:

a) $f(x) = \dfrac{x^3+1}{x+1}$

$x \neq -1$

$f(-1) = \frac{0}{0}$

$f(x) = \dfrac{\cancel{(x+1)}(x^2-x+1)}{\cancel{(x+1)}}$

removable at $x = -1$

$\lim_{x \to -1} (x^2 - x + 1)$

$= (-1)^2 - (-1) + 1$

$= 3$

$$\text{b) } f(x) = \frac{x^2+4x}{x^3-4x}$$

$$f(x) = \frac{\cancel{x}(x+4)}{\cancel{x}(x-2)(x+2)}$$

$$x \neq 0, \underbrace{2, -2}_{\text{infinite}}$$

$$\lim_{x \to 0} \frac{x+4}{(x-2)(x+2)}$$

$$= \frac{4}{-4} = -1$$

$x = 0$ removable.

c) $f(x)=\begin{cases} x^2, x \le 2 \\ 2x+1, x > 2 \end{cases}$

Test $x=2$

i) $f(2)=(2)^2=4$

ii) $\cancel{\lim_{x \to 2} f(x)}$ DNE

$\lim_{x \to 2^-} x^2$
$=(2)^2$
$=4$

$\lim_{x \to 2^+} 2x+1$
$=2(2)+1$
$=5$

$x=2$
jump
discont.

$$\text{d) } f(x) = \begin{cases} x^2 + 4, \text{ if } x \le 0 \\ x + 1, \text{ if } 0 < x \le 1 \\ x^2 + 1, \text{ if } x > 1 \end{cases}$$

Test $x = 0$

i) $f(0) = 0^2 + 4 = \cancel{6}\,4$

ii) $\lim_{x \to 0} f(x)$ DNE

$\lim_{x \to 0^+} x + 1 = 1$

$\lim_{x \to 0^-} x^2 + 4 = 4$

Jump at $x = 0$

Test $x=1$

i) $f(1) = 1+1 = 2$

ii) $\lim_{x \to 1} f(x) = 2$

$\lim_{x \to 1^-} x+1 = 2$

$\lim_{x \to 1^+} x^2+1 = 2$

iii) $f(1) = \lim_{x \to 1} f(x) = 2$

$\therefore$ cont $x=2$.

$$f(x) = \begin{cases} \sqrt{x+1} & 0 \le x \le 3 \\ 5-x & 3 < x \le 5 \end{cases}$$

Is f continuous at $x=3$? Prove

i) $f(3) = \sqrt{3+1} = 2$

ii) $\lim\limits_{x\to 3} f(x) = 2$

$\lim\limits_{x\to 3^-} \sqrt{x+1}$
$= 2$

$\lim\limits_{x\to 3^+} 5-x = 2$

iii) Since $f(3) = \lim_{x \to 3} f(x) = 2$

$\therefore$ cont $\underline{x = 3}$

e) $f(x)=\dfrac{|x^2-x-20|}{x-5}$

Jump

NPV

$x=5$

1) $f(5)=\dfrac{|5^2-5-20|}{5-5}=\dfrac{0}{0}$

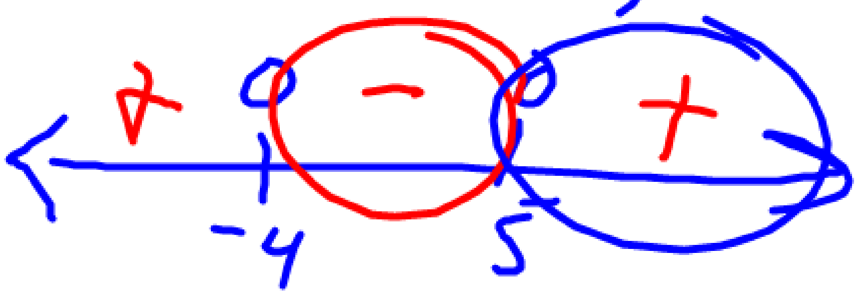
2) $\lim\limits_{x\to 5}\dfrac{|x^2-x-20|}{x-5}$

l_1

$$\lim_{x\to 5^-} \frac{-\cancel{(x-5)}(x+4)}{\cancel{x-5}} \qquad -9$$

$$\lim_{x\to 5^+} \frac{\cancel{(x-5)}(x+4)}{\cancel{(x-5)}} = 9$$

$(x-5)(x+4)$



$x = 5$

jump

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Oral Exercises #'s 6-10

Written Exercises

#'s 1, 2, ~~3-7~~, 8a, 10 b,c,e,g,h,j,k,l,m,o

3-5, 7,