3.3 Evaluating Limits

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Learning Targets:

SWBAT evaluate limits of a variety of functions using a number of diferent techniques.

1. Evaluating Limits By Direct Substitution

Ex.1 Find $\lim_{x \to 5}\left(x^2+2x-3\right)$

$= (5)^2 + 2(5) - 3$

$= 25 + 10 - 3$

$= 32$

Ex.2 Find $\lim_{x \to 1} \dfrac{x^4 - 5x^2 + 1}{x + 2}$

$$= \frac{(1)^4 - 5(1)^2 + 1}{1 + 2}$$

$$= \frac{-3}{3} = -1$$

Ex.3 Find $\lim_{x \to 3} \sqrt{x^2 + x}$

$= \sqrt{(3)^2 + 3}$

$= \sqrt{12}$

$= \sqrt{4 \cdot 3}$

$= 2\sqrt{3}$

$$\lim_{x\to 9} \frac{\log_3 x}{\sin\left(\frac{\pi x}{18}\right)}$$

$$= \frac{\log_3 9}{\sin\left(\frac{9\pi}{18}\right)} = \frac{\log_3 9}{\sin\left(\pi/2\right)} = \frac{\log_3 9}{1}$$

$$= 2$$

$\log_3 9 = x$

$3^x = 9$

$3^x = 3^2$

$\log_3 9$

2

$\log_3 3$

If our function is a **polynomial** we can use **direct substitution** to evaluate our limit. If our function is **rational or algebraic**, we can use direct substitution provided it does not make our function undefined.

2. Evaluating One Sided Limits

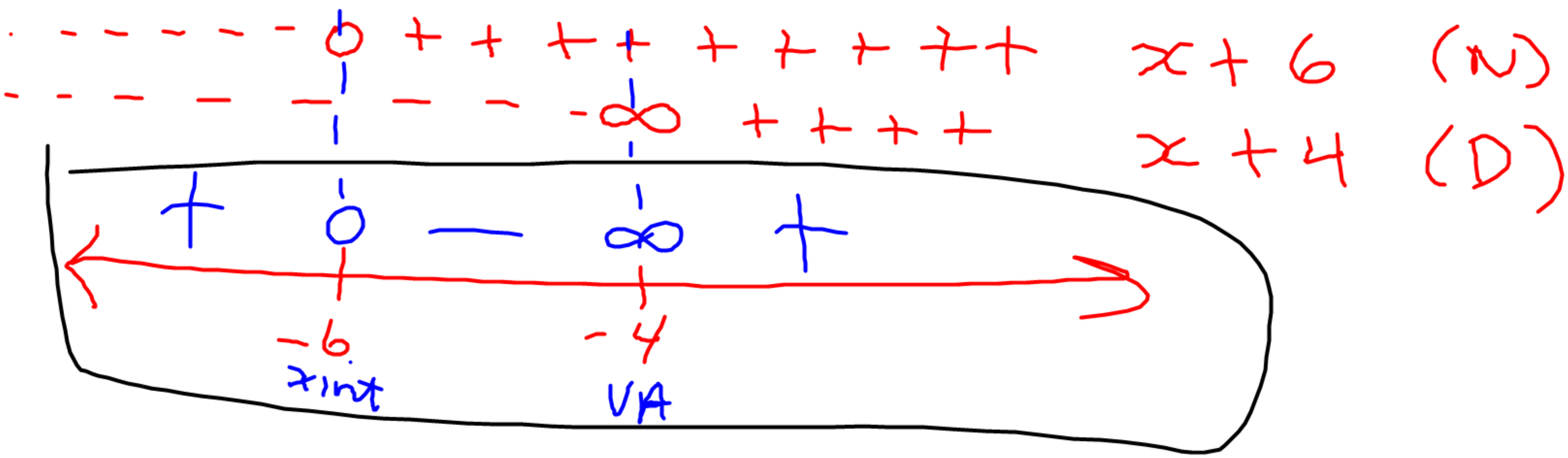
Ex.1 Determine each of the following limits:

a) $\lim_{x\to -4^{+}} \frac{x+6}{x+4}$ ∞

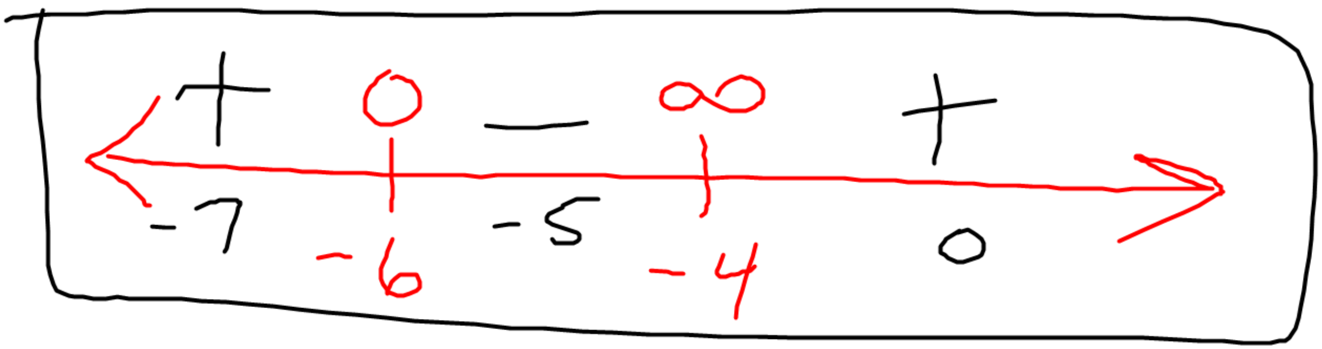
b) $\lim_{x\to -4^{-}} \frac{x+6}{x+4}$ $-\infty$

c) $\lim_{x\to -4} \frac{x+6}{x+4}$ DNE

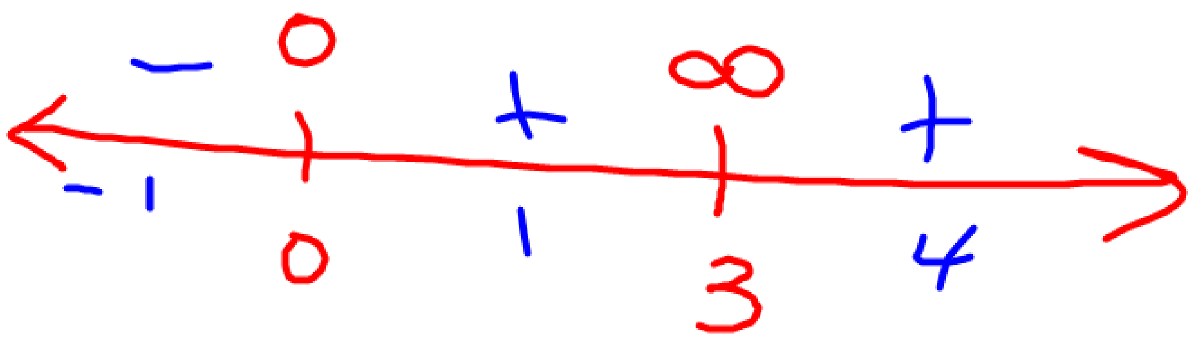
When we evaluate a limit at a VA, we use a sign chart.



$$\frac{x+6}{x+4}$$

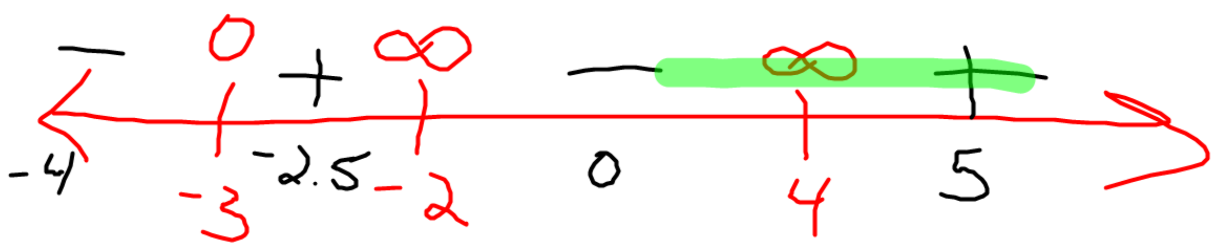


d) $\lim_{x \to 3^-} \frac{x}{(x-3)^2}$ ∞



$$\lim_{x \to 4} \frac{x+3}{x^2-2x-8} = \frac{7}{0} \quad S.C.$$

$$\lim_{x \to 4} \frac{x+3}{(x-4)(x+2)} \quad DNE$$



e) $$\lim_{x \to -5^-} \frac{x+2}{(x+5)^3}$$

Ex.2 Let $f(x)=\begin{cases} x^2, \text{if } x\le 1 \\ 2-x, \text{if } x>1 \end{cases}$ find the limits if they exist.

a) $\lim_{x\to 1^-} f(x)$

$\lim_{x\to 1^-} x^2 = (1)^2$

$= 1$

b) $\lim_{x\to 1^+} f(x)$

$\lim_{x\to 1^+} 2-x = 2-1$

$= 1$

c) $\lim_{x\to 1} f(x) = 1$

a) $\lim_{x\to -2} f(x)$

$\lim_{x\to -2} x^2 = (-2)^2$

$= 4$

b) $\lim_{x\to 5} f(x)$

$\lim_{x\to 5} 2-x = 2-5 = -3$

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#'s 1-7, 10 - 13, 16 - 18, 61 - 63

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$$g(x) = \begin{cases} \dfrac{x^2-4}{x+2} & x \neq -2 \\ -5 & \text{if } x = -2 \end{cases}$$

a) $\lim\limits_{x \to 3} \dfrac{(3)^2-4}{3+2} = 1$

b) $\lim\limits_{x \to -2} \dfrac{x^2-4}{x+2}$

(63)

$$h(x) = \begin{cases} 2 & x < 0 \\ x + 2 & 0 \le x \le 4 \\ x^2 - 11 & x > 4 \end{cases}$$

a) $\lim_{x \to -6} h(x) = 2$

b) $\lim_{x \to 0^-} h(x) = 2$

c) $\lim_{x \to 0^+} \cancel{h(x)}\, x + 2 = 2$

d) $\lim_{x \to 0} h(x) = 2$

$$\lim_{x \to 3} x + 2 = 5$$

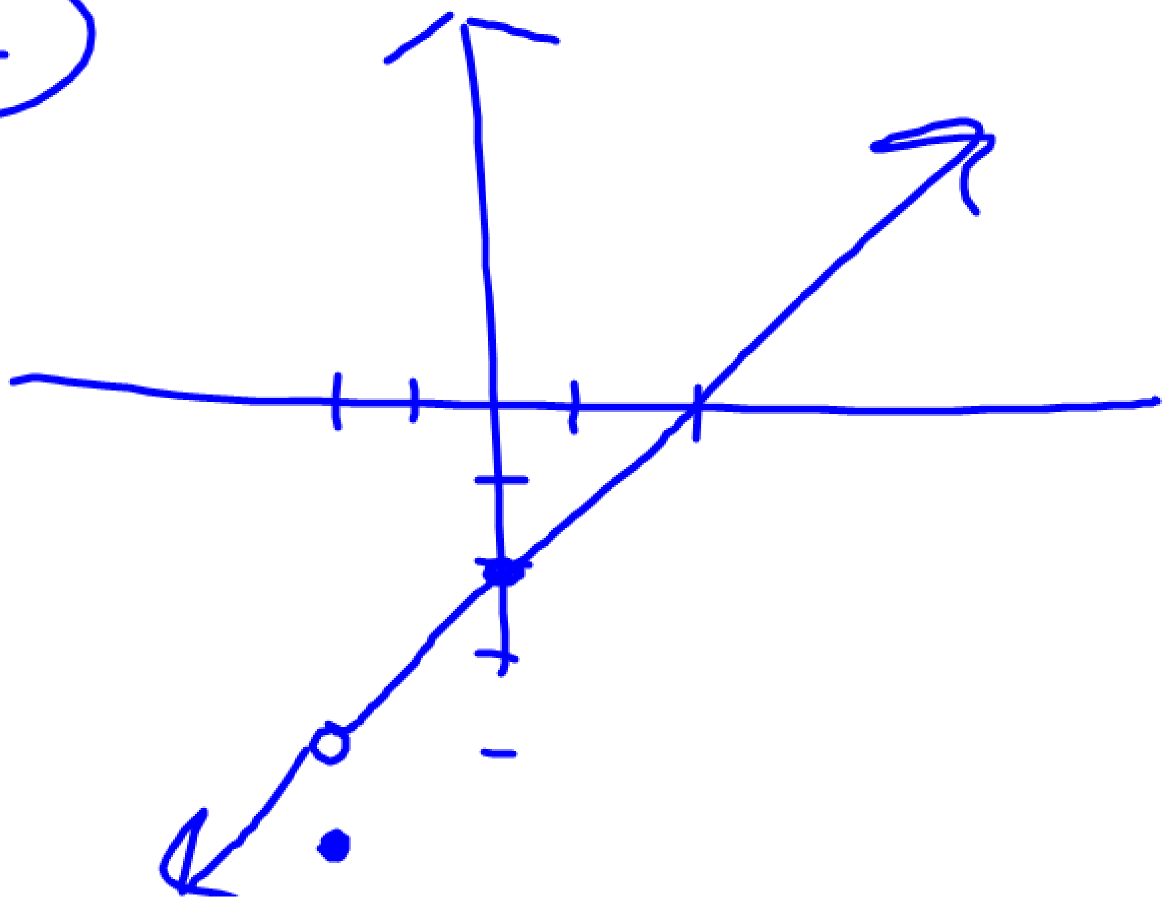
f) $$\lim_{x \to 4^-} x + 2 = 6$$

g) $$\lim_{x \to 4^+} x^2 - 11 = 16 - 11 = 5$$

h) $$\lim_{x \to 4} h(x) \text{ DNE}$$

$$\frac{x^2-4}{x+2} = \frac{(x-2)\cancel{(x+2)}}{\cancel{(x+2)}}$$

$x \neq -2$



3. Evaluating Limits By Factoring First

Ex.3 Determine each of the following limits:

a) $\lim_{x\to 6} \frac{x^2 - 7x + 6}{x^2 - 36}$

Direct sub. results in $\frac{0}{0}$.

$\lim_{x\to 6} \frac{\cancel{(x-6)}(x-1)}{\cancel{(x-6)}(x+6)}$

$\lim_{x\to 6} \frac{x-1}{x+6} = \frac{6-1}{6+6} = \frac{5}{12}$

b) $\lim_{x\to 2} \frac{x^3 - 8}{x^2 - 3x + 2}$

$\lim_{x\to 2} \frac{\cancel{(x-2)}(x^2+2x+4)}{\cancel{(x-2)}(x-1)}$

$\frac{(2)^2+2(2)+4}{2-1}$

$= 12$

4. Evaluating Limits By Simplifying

Ex.4 Determine each of the following limits:

a) $\lim\limits_{h\to 0}\dfrac{(2+h)^2-4}{h}$

$$\frac{\left(\cancel{(2+h)}-2\right)(2+h+2)}{\cancel{h}}$$

$$\lim_{h\to 0}\frac{\cancel{4}+(4h+h^2)\cancel{-4}}{h}$$

$$=\lim_{h\to 0}\frac{\cancel{h}(4+h)}{\cancel{h}}=4+0=4$$

b) $\displaystyle \lim_{x\to 0} \frac{\frac{3}{3}\cdot\frac{1}{x+3} - \frac{1}{3}\cdot\frac{(x+3)}{(x+3)}}{2x}$

$$\lim_{x\to 0} \frac{\frac{3}{3(x+3)} - \frac{(x+3)}{3(x+3)}}{2x}$$

$$\lim_{x\to 0} \frac{\frac{3-x-3}{3(x+3)}}{2x}$$

$$\lim_{x\to 0} \frac{\left(\frac{-x}{3(x+3)}\right)}{\left(\frac{2x}{1}\right)} = \lim_{x\to 0} \frac{-\cancel{x}}{3(x+3)} \cdot \frac{1}{2\cancel{x}}$$

$$= \frac{-1}{3(0+3)} \cdot \frac{1}{2} = \frac{-1}{18}$$

5. Evaluating Limits By Rationalizing

Binomial Radical
Rationalize

a) $\lim\limits_{x\to 0} \frac{\sqrt{x+1}-1}{x}$

$$\lim_{x\to 0} \frac{(\sqrt{x+1}-1)}{x} \cdot \frac{(\sqrt{x+1}+1)}{(\sqrt{x+1}+1)}$$

only foil out conjugates

$$\lim_{x\to 0} \frac{\cancel{x}\ 1}{\cancel{x}(\sqrt{x+1}+1)} = \frac{1}{\sqrt{0+1}+1} = \frac{1}{2}$$

$$\text{b) } \lim_{x\to 6} \frac{\sqrt{3+r}-3}{r-6}$$

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#'s 19 - 26, 28, 29, 31 - 38

(62b)

(26)

$$\lim_{t \to -1} \frac{t^4 - 3t^2 + 2}{t+1}$$

$$\lim_{t \to -1} \frac{(t^2-2)(t^2-1)}{t+1}$$

$$\lim_{t \to -1} \frac{(t^2-2)(t-1)\cancel{(t+1)}}{\cancel{(t+1)}}$$

$$= \left((-1)^2 - 2\right)(-1-1)$$

$$= (-1)(-2) = 2$$

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$$\lim_{x \to 0} \frac{(x+1)^2 - 1}{x}$$

$$\lim_{x \to 0} \frac{x^2 + 2x}{x}$$

$$\lim_{x \to 0} \frac{\cancel{x}(x+2)}{\cancel{x}} = 2$$

$$\frac{(\cancel{x+1-1})(x+1+1)}{\cancel{x}}$$

$$\lim_{x \to 3} \frac{\cancel{(3-x)}(-1)}{\cancel{(x-3)}(x+2)}$$

(32)

$$\lim_{x\to 1} \frac{\frac{3}{3}\cdot\frac{1}{(2x+1)} - \frac{1}{3}\cdot\frac{(2x+1)}{(2x+1)}}{\left(\frac{x-1}{1}\right)}$$

$$1-x$$
$$= -1(x-1)$$

$$\lim_{x\to 1} \frac{\frac{3}{3(2x+1)} - \frac{(2x+1)}{3(2x+1)}}{\left(\frac{x-1}{1}\right)}$$

$$\lim_{x\to 1} \frac{3-2x-1}{3(2x+1)} \cdot \frac{1}{(x-1)}$$

$$\lim_{x\to 1} \frac{2-2x}{3(2x+1)} \cdot \frac{1}{(x-1)} = \lim_{x\to 1} \frac{2(\cancel{1-x})}{3(2x+1)} \cdot \frac{-1}{\cancel{x-1}}$$

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$$\lim_{y\to 0} \frac{\left(\sqrt{y+2} - \sqrt{2}\right)}{y} \cdot \frac{\left(\sqrt{y+2} + \sqrt{2}\right)}{\left(\sqrt{y+2} + \sqrt{2}\right)}$$

$$\lim_{y\to 0} \frac{\cancel{y} \cancel{+2 - 2}}{\cancel{y}\left(\sqrt{y+2} + \sqrt{2}\right)}$$

$$\lim_{y\to 0} \frac{1}{\sqrt{y+2} + \sqrt{2}}$$

$$\frac{1}{\sqrt{2} + \sqrt{2}} = \frac{1}{2\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{4}$$

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$$\lim_{x \to 0} \frac{(x+1)^2 - 1}{x}$$

$$\lim_{x \to 0} \frac{\cancel{(x+1-1)}(x+1+1)}{\cancel{x}}$$

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$$\lim_{x \to 4} \frac{\frac{2}{2}\cdot\frac{1}{x-2} - \frac{1}{2}\cdot\frac{(x-2)}{(x-2)}}{x-4}$$

$$\lim_{x \to 4} \frac{\frac{2}{2(x-2)} - \frac{(x-2)}{2(x-2)}}{(x-4)}$$

$$\lim_{x \to 4} 2 - x$$

32

$$\lim_{x \to 1} \frac{\frac{3}{3}\cdot\frac{1}{2x+1} - \frac{1}{3}\cdot\frac{2x+1}{2x+1}}{x-1}$$

$$\lim_{x \to 1} \left[\frac{3}{3(2x+1)} - \frac{(2x+1)}{3(2x+1)} \right] \frac{1}{x-1}$$

$$\lim_{x \to 1} \frac{2-2x}{3(2x+1)} \cdot \frac{1}{x-1}$$

$$\lim_{x \to 1} \frac{-2\cancel{(-1+x)}}{3(2x+1)} \cdot \frac{1}{\cancel{x-1}}$$

$$-2/9$$

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$$\lim_{y \to 0} \frac{\left(\sqrt{y+2} - \sqrt{2}\right)}{y} \cdot \frac{\left(\sqrt{y+2} + \sqrt{2}\right)}{\left(\sqrt{y+2} + \sqrt{2}\right)}$$

6. Evaluating Limits At Infinity Of Rational Functions

6. Evaluating Limit… Rational Fu…

Evaluate the following limits:

a) $\lim\limits_{x\to\infty} \dfrac{6x-5}{2x+3} \cdot \dfrac{\frac{1}{x}}{\frac{1}{x}}$

$$\lim_{x\to\infty} \frac{6-\frac{5}{x}}{2+\frac{3}{x}} = \frac{6}{2} = 3$$

If the degree of the numerator = degree of the denominator the limit is the ratio of the leading coefficients.

$\lim\limits_{x\to\pm\infty}$

b) $\lim\limits_{x\to-\infty} \dfrac{x^2+1}{2x^2+3x-1}$ $= \dfrac{1}{2}$

c) $\lim\limits_{x \to \infty} \dfrac{x+7}{2x^2+3x-1}$ $\dfrac{\frac{1}{x^2}}{\frac{1}{x^2}}$

$$\lim_{x \to \infty} \frac{\frac{1}{x} + \frac{7}{x^2}}{2 + \frac{3}{x} - \frac{1}{x^2}} = \frac{0}{2} = 0$$

If the degree of the numerator < degree of the denominator the limit is zero.

d) $\lim\limits_{x\to\infty} \dfrac{x^2+5x}{x-5}$ $\quad \dfrac{1}{x}$ $\quad \dfrac{1}{x}$

$$\lim_{x\to\infty} \frac{x+5}{1-\frac{5}{x}} \quad \infty$$

If the degree of the numerator > degree of the denominator the limit will be either positive or negative infinity.

7. Limits At Infinity Of Functions Containing Radicals

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Non Rationals

$$\lim_{x \to -\infty} \frac{\sqrt{9x^2 + 2x}}{x - 3}$$

$$\lim_{x \to -\infty} \frac{\sqrt{x^2\left(9 + \frac{2}{x}\right)}}{x\left(1 - \frac{3}{x}\right)}$$

$$\lim_{x \to -\infty} \frac{\sqrt{9x^2}}{x}$$

$$\lim_{x \to -\infty} \frac{3\sqrt{x^2}}{x}$$

$$\sqrt{x^2} = \pm x = |x|$$

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

$$\lim_{x\to-\infty} \frac{3|x|}{x} = \frac{-3\cancel{x}}{\cancel{x}} = -3$$

$$\lim_{x\to\infty} \frac{\sqrt{16x^2+4}}{3-x}$$

$$\lim_{x\to\infty} \frac{\sqrt{x^2\left(16+\frac{4}{x^2}\right)}}{x\left(\frac{3}{x}-1\right)} = \lim_{x\to\infty} \frac{\sqrt{16x^2}}{-x}$$

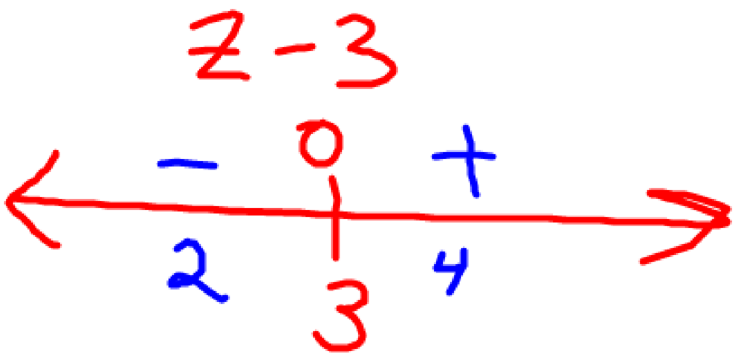
$$\lim_{x \to \infty} \frac{4\sqrt{x^2}}{-x} = \frac{4|x|}{-x} = \frac{4x}{-x} = -4$$

8. Limits Involving Absolute Value

Evaluate the following limits:

a) $\lim\limits_{x\to 3^-} \dfrac{z-3}{|z-3|}$

$$|z-3| = \begin{cases} z-3 \\ -(z-3) \end{cases}$$



$$\lim_{z\to 3^-} \frac{\overset{1}{\cancel{z-3}}}{-(\cancel{z-3})} = -1$$

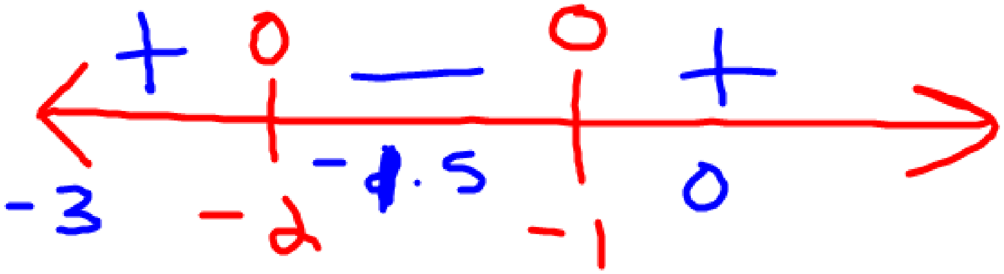
b) $\lim_{x \to -2^{+}} \dfrac{\left|x^2 + 3x + 2\right|}{x + 2}$

c) $\lim\limits_{x\to -2^-} \dfrac{|x^2+3x+2|}{x+2}$

$$= \lim_{x\to -2^-} \frac{\cancel{(x+2)}(x+1)}{\cancel{(x+2)}}$$

$(x+2)(x+1)$

$$= -2+1 = -1$$



d) $\lim_{x \to -2} \dfrac{\left|x^2 + 3x + 2\right|}{x + 2}$

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