Unit 3: Volumes of Revolution and Cross Sectional Areas

3.1 The Disk Method

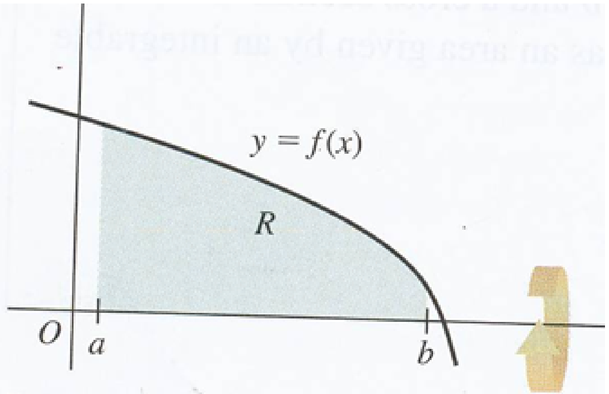


FIGURE 6.26

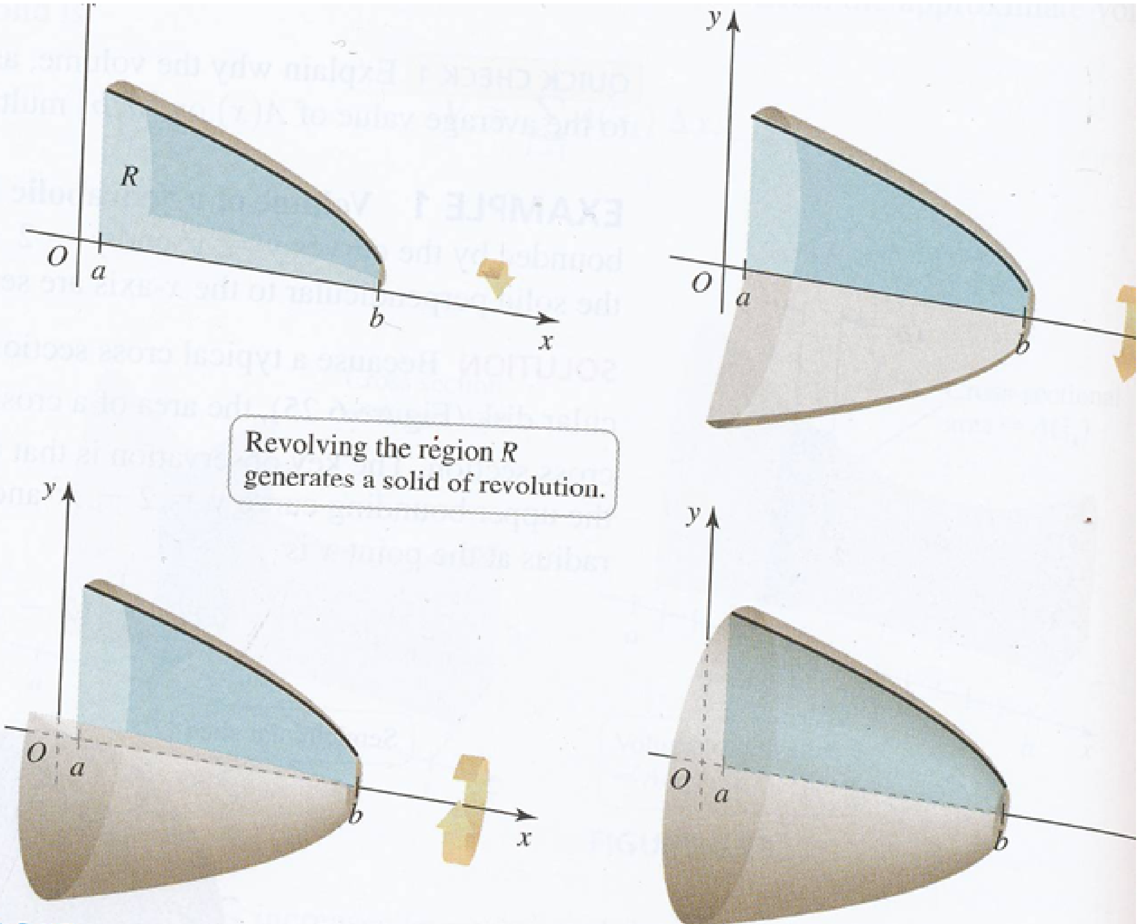
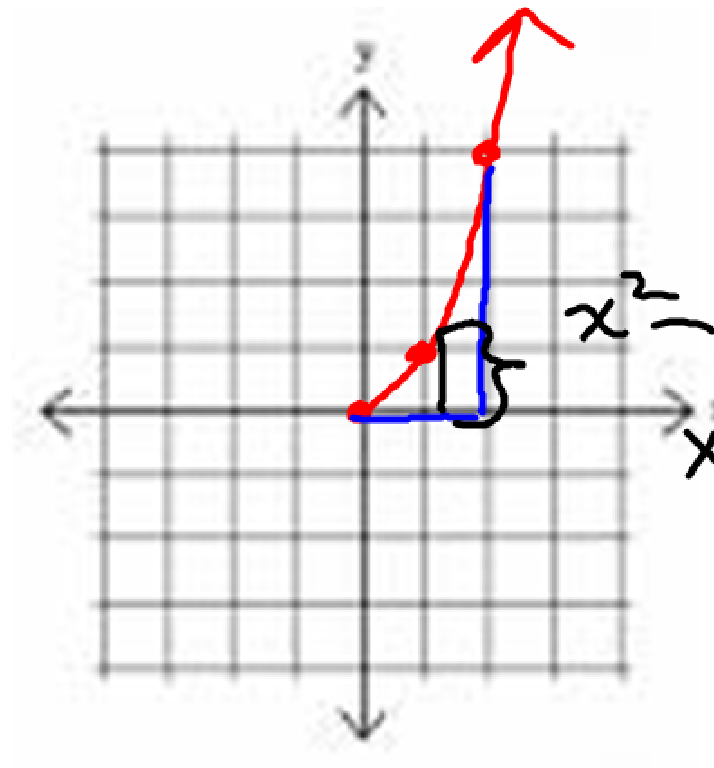

R
O
a
b
x
y
Revolving the region R generates a solid of revolution.

FIGURE 6.27

Say we have the curve $y=x^2$ from $x=0$ to $x=2$. If we let R be the region bound by this curve and the x-axis, and we revolve this region about the x-axis, find the volume of the shape formed. (See visual next slide)

$$V=\int_0^2 \pi r^2\,dx$$



$$x^2-0=r$$

$$x^2=r$$

$$=\pi\int_0^2 (x^2)^2\,dx$$

$$=\pi\int_0^2 x^4\,dx=\frac{32\pi}{5}$$

In general, the volume of a solid of known integrable cross section area $A(x)$ from $x = a$ to $x = b$ is given by: $V = \int_a^b A(x)dx$.

When our cross sectional area touches the axis of rotation we are using what is called the disk method.

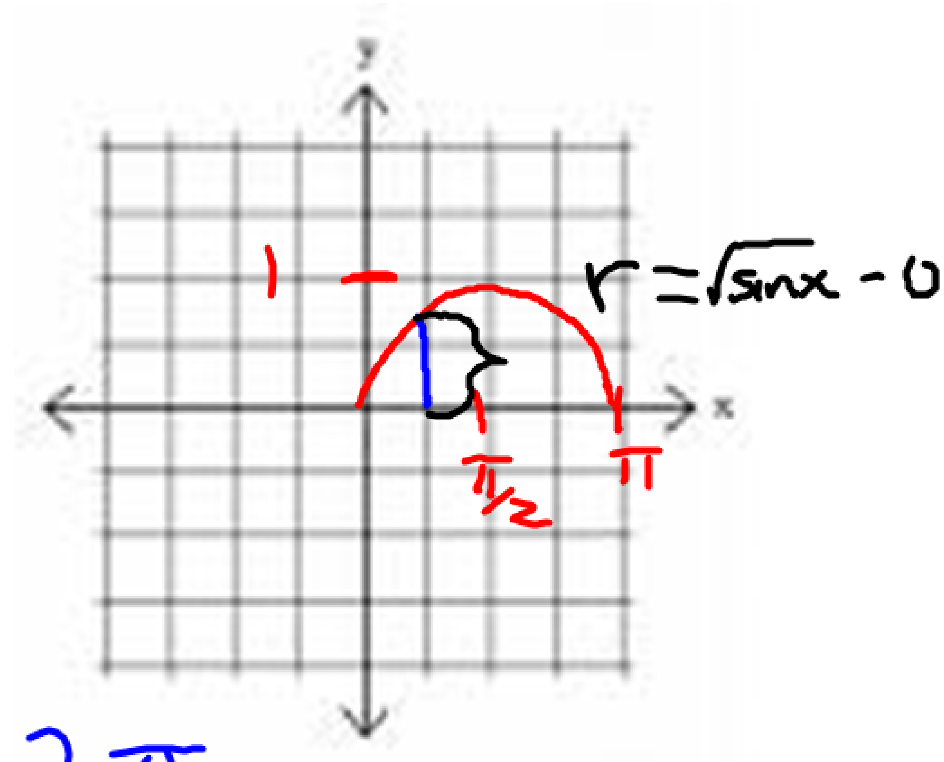
Ex.1 Find the volume of a solid formed by revolving the region bounded by the graph of $f(x)=\sqrt{\sin x}$ and the axis from $0 \le x \le \pi$ about the x-axis.

(A calculator may be used.)

$$V=\int_0^{\pi} \pi r^2\,dx$$

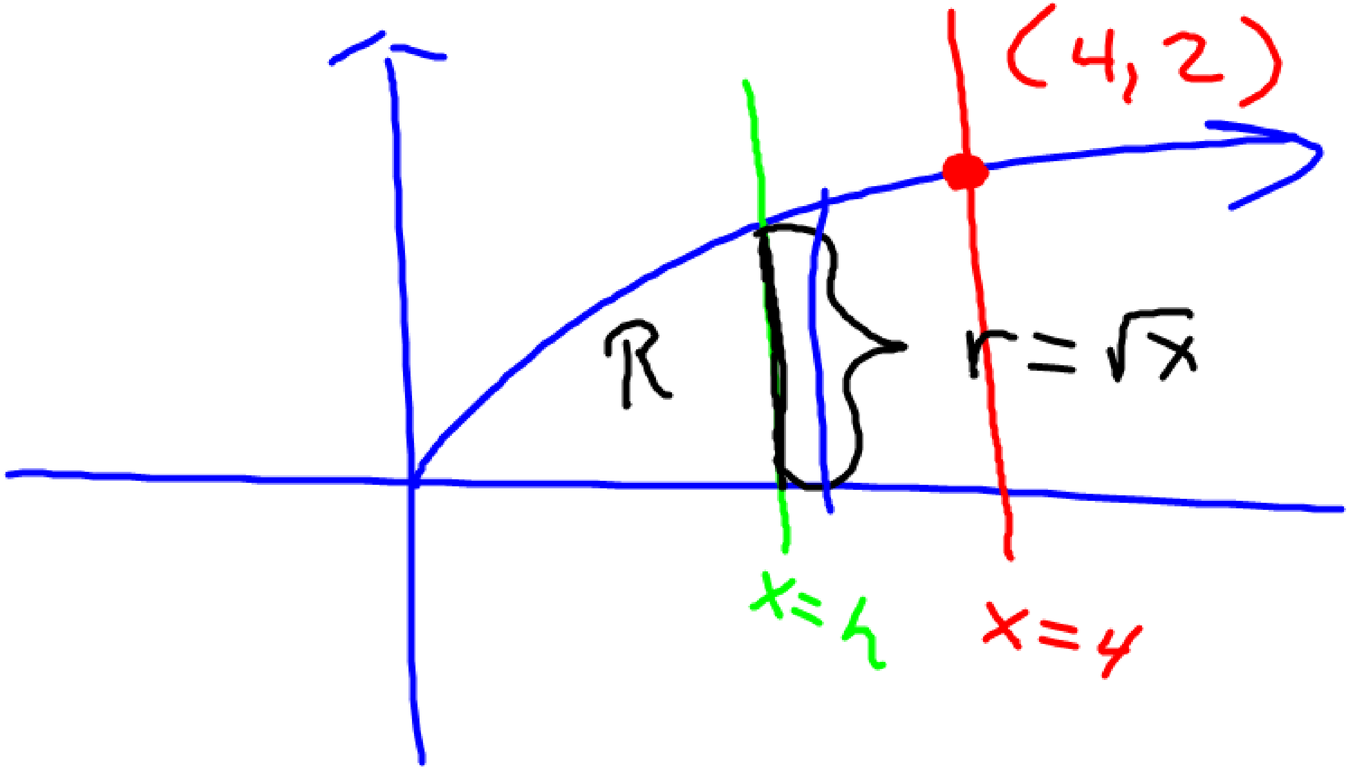
$$V=\pi\int_0^{\pi}\left(\sqrt{\sin x}\right)^2 dx$$

$$=\pi\int_0^{\pi}\sin x\,dx = 2\pi$$



Example 2 1998 AP Calculus AB Scoring Guidelines

1. Let R be the region bounded by the x–axis, the graph of $y = \sqrt{x}$, and the line $x = 4$.
   (a) Find the area of the region R.
   (b) Find the value of h such that the vertical line $x = h$ divides the region R into two regions of equal area.
   (c) Find the volume of the solid generated when R is revolved about the x–axis.
   (d) The vertical line $x = k$ divides the region R into two regions such that when these two regions are revolved about the x–axis, they generate solids with equal volumes. Find the value of k.



a) $\int_0^4 \sqrt{x}\, dx = \frac{16}{3}$

b) $\int_0^h \sqrt{x}\, dx = \frac{8}{3}$

$\frac{2}{3} x^{3/2} \Big|_0^h = \frac{8}{3}$

$\frac{2}{3} h^{3/2} - 0 = 8/3$

$$\cancel{\frac{2}{3}} h^{3/2} = \cancel{\frac{8}{3}}$$

$$\left(h^{3/2}\right)^2 = (4)^2$$

$$h^3 = 16$$

$$h = \sqrt[3]{16}$$

c)

$$V = \int_0^4 \pi \left(\sqrt{x}\right)^2 dx$$

$$= \pi \int_0^4 x \, dx = 8\pi$$

d)

$$\cancel{\pi} \int_0^k \left(\sqrt{x}\right)^2 dx = 4\cancel{\pi}$$

$$\int_0^k x \, dx = 4$$

$$\left.\frac{x^2}{2}\right|_0^k = 4$$

$$\frac{k^2}{2} - 0 = 4$$

$$k^2 = 8$$

$$k = \pm\sqrt{8}$$

$$k = \sqrt{8}$$

Example 2 **1998 AP Calculus AB Scoring Guidelines**

1. Let R be the region bounded by the x–axis, the graph of $y = \sqrt{x}$, and the line $x = 4$.
   (a) Find the area of the region R.
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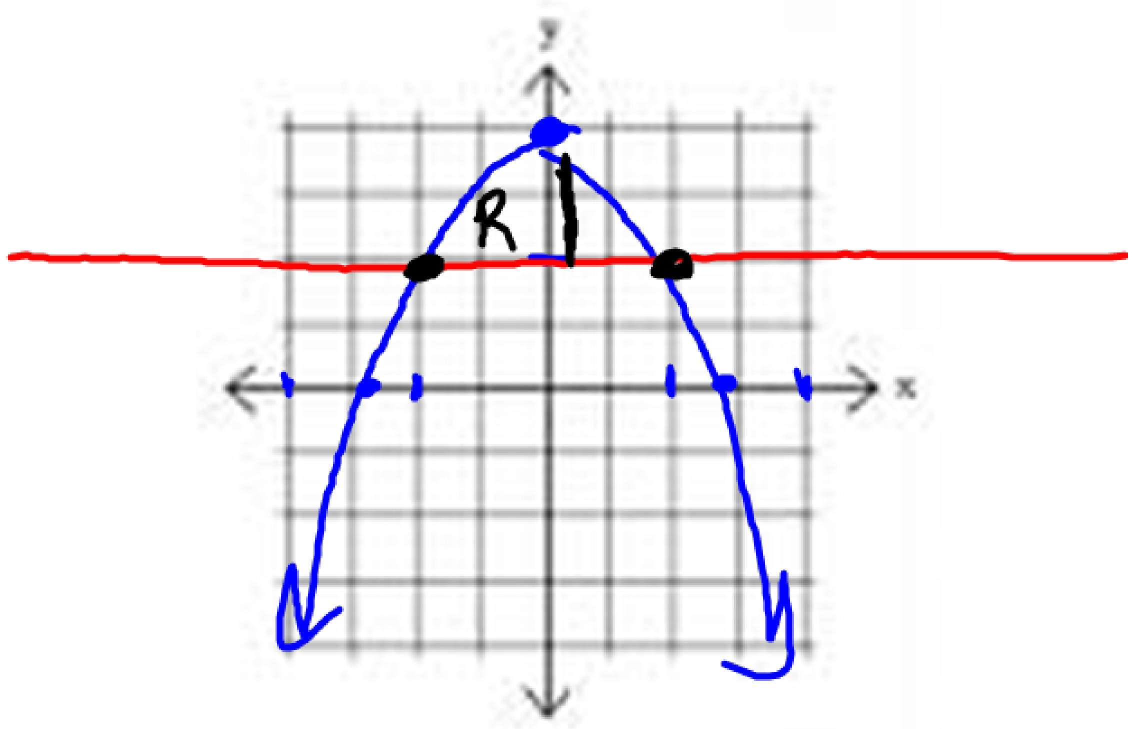
Example 3 Find the volume of the solid formed by revolving the region bounded by $f(x) = 2 - x^2$ and $g(x) = 1$ about the line $y = 1$.

$r = 2 - x^2 - 1$

$r = 1 - x^2$



Int Points

$2 - x^2 = 1$

$0 = x^2 - 1$

$1 = x^2$

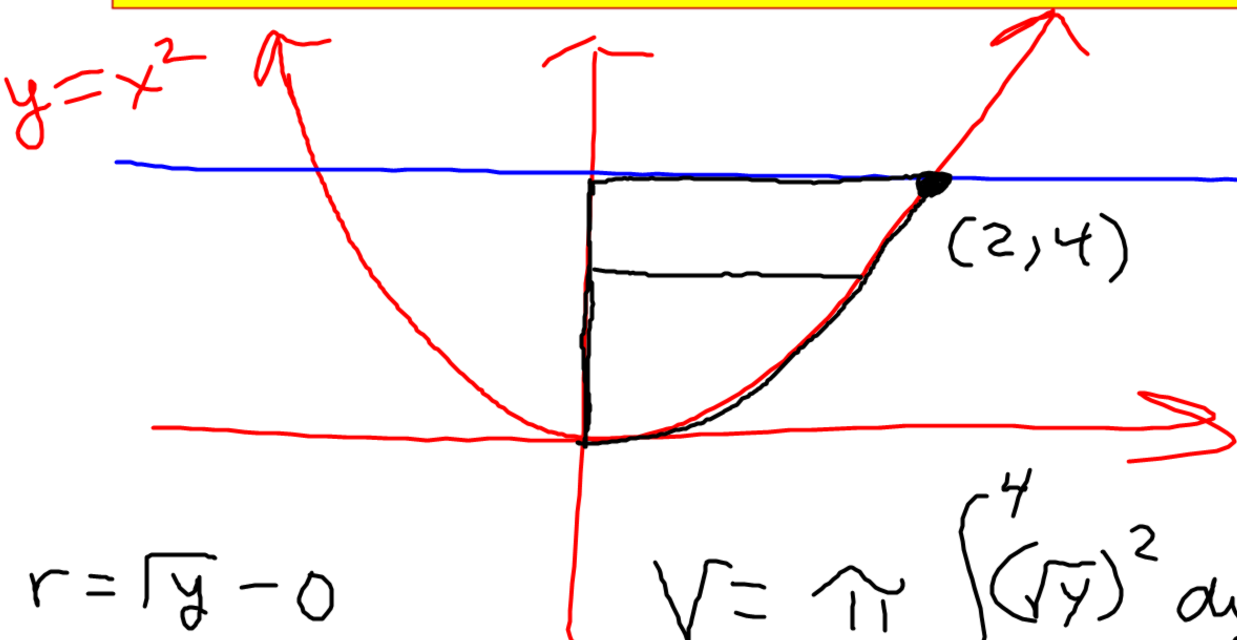
$\pm 1 = x$

$$V = \pi \int_{-1}^{1} (1 - x^2)^2 \, dx$$

$$= \frac{16}{15}\pi$$

Disk Method About the Y-Axis

Example: Let R be the region in the first quadrant bounded by the curve $y = x^2$ and $y = 4$. Find the volume of this solid if region R is revolved about the y-axis



$y = x^2$

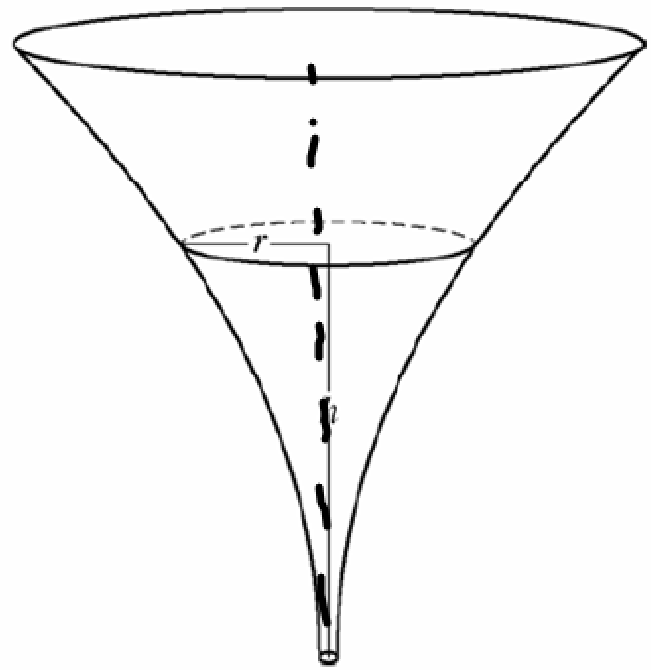
$\pm\sqrt{y} = x$

$r = \sqrt{y} - 0$

$r = \sqrt{y}$

$$V = \pi \int_0^4 (\sqrt{y})^2\, dy = \pi \int_0^4 y\, dy = 8\pi$$

Question 5



The inside of a funnel of height 10 inches has circular cross sections, as shown in the figure above. At height h, the radius of the funnel is given by $r = \frac{1}{20}\left(3 + h^2\right)$, where $0 \le h \le 10$. The units of r and h are inches.

(a) Find the average value of the radius of the funnel.

(b) Find the volume of the funnel.

(c) The funnel contains liquid that is draining from the bottom. At the instant when the height of the liquid is $h = 3$ inches, the radius of the surface of the liquid is decreasing at a rate of $\frac{1}{5}$ inch per second. At this instant, what is the rate of change of the height of the liquid with respect to time?

a)

ave value

$$= \frac{1}{10-0} \int_0^{10} \frac{1}{20}(3+h^2)\,dh$$

$$= \frac{1}{200} \int_0^{10} (3+h^2)\,dh$$

$$= \frac{1}{200}\left[3h + \frac{h^3}{3} \Big|_0^{10} \right]$$

$$= \frac{1}{200}\left[30 + \frac{1000}{3} \right]$$

b) $$V = \pi \int_0^{10} \left(\frac{1}{20}(3+h^2)\right)^2 dh$$

$$V = \pi \int_0^{10} \frac{1}{400}\left[9 + 6h^2 + h^4\right] dh$$

$$= \frac{\pi}{400}\left[\left. 9h + 2h^3 + \frac{h^5}{5} \right|_0^{10}\right]$$

$$= \frac{\pi}{400}\left[9(10) + 2(10)^3 + \frac{10^5}{5}\right] in^3$$

c) $\frac{dr}{dt} = -\frac{1}{5}$ in/s

$\frac{dh}{dt} = ?$

$h = 3$ in

$r = \frac{1}{20}(3+h^2)$

$\frac{dr}{dt} = \frac{1}{20}\left(2h\frac{dh}{dt}\right)$

$-\frac{1}{5} = \frac{1}{20}(2(3))\,dh/dt$

$-\frac{2}{3}$ in/s $= \left(\frac{20}{6}\right)\left(-\frac{1}{5}\right) = dh/dt$

Assignment
Page 392
#'s 17,18,19,20
35b,c
(about the y axis)