# 2.5 Exponential Growth and Decay

## Learning Targets:

1. SWBAT use their knowledge of solving differential equations to solve exponential growth problems.

2. SWBAT use their knowledge of solving differential equations to solve exponential decay problems.

In Pre-calculus 30 you may have used these equations when studying Ecponential growth and decay.

$$N = N_0(2)^{\frac{t}{d}}$$

$$M = M_0(2)^{-\frac{t}{h}}$$

$4(2)^{16/4}$

**Law of Exponential Change**

Suppose we are interested in a quantity $y$ that increases or decreases at a rate proportional to the amount present. If we know the amount present at $t = 0$, say $y_0$, we can find $y$ as a function of time by solving the following initial value problem given:

$$\frac{dy}{dt} = ky \quad \text{and} \quad \boxed{y = y_0 \text{ when } t = 0}$$

$$\frac{dy}{dt} = ky$$

$$\int \frac{dy}{y} = \int k\,dt$$

$$\ln|y| = kt + C$$

$$e \qquad\qquad e$$

$$|y| = e^{kt} \cdot e^{c}$$

let $\pm e^{c} = A$

$$y = A e^{kt}$$

$$y_0 = A e^{k(0)}$$

$$y_0 = A$$

exp. growth $k > 0$

$$\boxed{y = y_0 e^{kt}}$$

$k < 0$ exp. decay

Ex. 1. The change N, in the number of bacteria in a culture dish at time t is given by $\frac{dN}{dt} = 2N$. If $N = 3$ when $t = 0$, the approximate value of t when $N =$ 1[redacted] is:

1210.

a) 2 b) 3 c) 4 d) 5 e) 6

$$y = y_0 e^{2t}$$

$$y = 3e^{2t}$$

$$1210 = 3e^{2t}$$

$$\left(\frac{1210}{3}\right) = e^{2t}$$

$$\ln\left(\frac{1210}{3}\right) = \ln e^{2t}$$

$$\frac{1}{2}\ln\left(\frac{1210}{3}\right) = t$$

$$t \simeq 3 \text{ years}$$

Ex. 2 The number of bacteria in a culture grow at a rate of $1500e^{\frac{3t}{4}}$ per unit time. At $t=0$, the number of bacteria present was 2000. Find the number present at $t=4$

a) $2000e^3$ b) $6000e^3$ c) $2000e^6$ d) $1500e^6$ e) $1500e^3+500$

$$2000+\int_0^4 1500e^{3t/4}$$

Ex. 3 A colony of bacteria grown under ideal conditions grows exponentially with time. At the end of 7 hours there are 15000 bacteria. At the end of 12 hours there are 28000 bacteria present. How many bacteria were there initially?

$$y = y_0 e^{kt}$$

$$15000 = y_0 e^{7k}$$

$$\frac{15000}{e^{7k}} = y_0$$

$$28000 = y_0 e^{12k}$$

$$\frac{28000}{e^{12k}} = y_0$$

$$28000 = 15000e^{5k}$$

$$\frac{28}{15} = e^{5k}$$

$$\frac{1}{5}\ln\left(\frac{28}{15}\right) = k$$

$$y = y_0 e^{\frac{1}{5}\left(\ln\left(\frac{28}{15}\right)\right)t}$$

$$\frac{15000}{e^{7K}} = \frac{28000}{e^{12K}}$$

$$\frac{e^{12K}}{e^{7K}} = \frac{28\cancel{000}}{15\cancel{000}}$$

$$e^{5K} = \frac{28}{15}$$

$$\ln\left(e^{5K}\right) = \ln\left(\frac{28}{15}\right)$$

$$\cancel{5}K = \frac{1}{5}\ln\left(\frac{28}{15}\right)$$

$$y_0 = \frac{15000}{e^{7\left(\frac{1}{5}\ln\left(28/15\right)\right)}}$$

$$\approx 6260 \text{ cells} \approx y_0$$

Ex. 4 The concentration of a medication injected into a bloodstream drops at a rate proportional to the existing concentration. If the factor of proportionality is 30% per hour, in how many hours will the concentration be 1/10 of the initial concentration?

$k = -.3$

$y_0 = 1$

$y = 1/10$

$t = ?$

$y = y_0 e^{kt}$

$\frac{1}{10} = 1e^{(-.3t)}$

$\ln\left(\frac{1}{10}\right) = \ln\left(e^{-.3t}\right)$

$\frac{\ln(1/10)}{-.3} = \cancel{-.3}\,t$

$7.675h = t$

Ex. 5 A cup of coffee at $180^\circ$ F is placed on a table in a room at $68^\circ$ F. The differential equation for its temperature at time t is $\frac{dy}{dt} = -0.11(y - 68)$ with the initial condition $y(0) = 180$. After 10 minutes the temperature of the coffee is:

a) $96^\circ F$ b) $100^\circ F$ c) $105^\circ F$ d) $110^\circ F$ e) $115^\circ F$

Ex.6 The population grows according to the equation $\frac{dy}{dt} = ky$, where $k$ is the constant of proportionality and $t$ is measured in years. If the population doubles every 10 years, then find the value of $k$.

$$y = y_0 e^{kt}$$

$$2 = 1 e^{10k}$$

$$\frac{(\ln 2)}{10} = k = \underline{0.0693}$$

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A bacteria culture initially contains 100 cells and grows at a rate proportional to its size. after an hour the population has increased to 420.

a) find an expression for the number of bacteria after t hours

b) find the # of bacteria after 3 hours

c) find the rate of growth after 3 hours

d) when will the population reach 10,000?

More Video Examples

## Newton's Law of Cooling

- The rate of cooling of an object is proportional to the temperature difference between the object and its surroundings, provided that this difference is not too large. (This law also applies to warming).

  If $T(t)$ is the temperature of the object at time $t$ and $T_s$ is the temperature of the surroundings, then

  $$\frac{dT}{dt} = k(T - T_s)$$

  where $k$ is a constant.

$$T - T_s = (T_0 - T_s)e^{kt}$$

## Example

When a cold drink is taken from a refrigerator, its temperature is 5°C. After 25 minutes in a 20°C room its temperature has increased to 10°C. What is the temperature of the drink after $t$ minutes?

$T_0 = 5°c$

$T_s = 20°c$

$T = ?$

$$T - 20 = (5-20)e^{kt}$$

$$T = -15e^{kt} + 20$$

$$10 = -15e^{25(k)} + 20$$

$$-10 = -15e^{25k}$$

$$2/3 = e^{25k}$$

$$\ln\left(\frac{2}{3}\right) = 25K$$

$$\frac{1}{25}\ln\left(2/3\right) = K$$

2014 Multiple Choice Calculator Portion

81. At time $t = 0$ years, a forest preserve has a population of 1500 deer. If the rate of growth of the population is modeled by $R(t) = 2000e^{0.23t}$ deer per year, what is the population at time $t = 3$?

(A) 3987 (B) 5487 (C) 8641 (D) 10,141 (E) 12,628

$$1500 + \int_0^3 2000e^{0.23t}\,dt$$

We said earlier if $y = y_0 e^{kt}$ and $k$ is negative, this represents exponential decay.

**Half Life** of a radioactive substance is the time required for a radioactive element to decay to half of its original amount.

$H = \dfrac{\ln 2}{k}$ where H is the half life and $k$ is the constant of proportionality.

Ex. 7 Scientists who do Carbon 14 dating use 5700 years for its half life. Find the age of a sample in which 10% of the radioactive material originally present has decayed.

$$5700 = \frac{\ln 2}{k}$$

$$k = \frac{-\ln 2}{5700}$$

$$y = y_0 e^{kt}$$

$$.9 = 1\left(e^{\frac{-\ln 2}{5700}t}\right)$$

$$\ln(.9) = \left(\frac{-\ln 2}{5700}\right)t$$

$$\frac{5700(\ln .9)}{-\ln 2} = t$$

$$866.418 \text{ years} = t$$

Assignment

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#'1-4,12-16,18-20,28,29

28

$$\boxed{\frac{dy}{dt} = -0.6y}$$

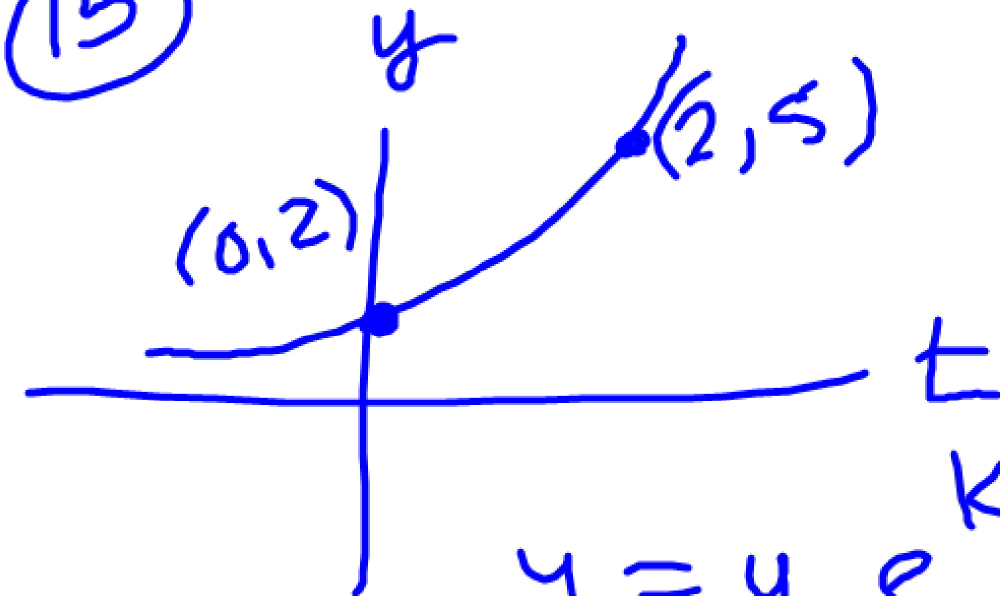
$t=0 \longrightarrow$ 100 grams

$t=1 \longrightarrow$ ? grams

$y = y_0 e^{-0.6t}$

$y = 100e^{-0.6t}$

15




$$y = y_0 e^{kt}$$

$$y = 2e^{kt}$$

$$5 = 2e^{2k}$$

③ $y(0) = 50$

$y(5) = 100$

$$\frac{dy}{dt} = ky$$

$$y = y_0 e^{kt}$$

$$y = 50e^{kt} \longrightarrow y = 50e^{\frac{\ln 2}{5} t}$$

$$100 = 50e^{5k}$$

$$2 = e^{5k}$$

$$\frac{\ln 2}{5} = k$$

$$A = A_0 e^{rt}$$

$$A = P\left(1 + \frac{i}{n}\right)^{nt}$$