2.4 Rates of Change and Tangent Lines

We encounter average rate changes all the time:

- Average speed
- Average growth rates
- Average monthly rainfall

Average rate change of a quantity over a period of time is the amount of change divided by the time it takes.

In other words we are talking about **slope**.

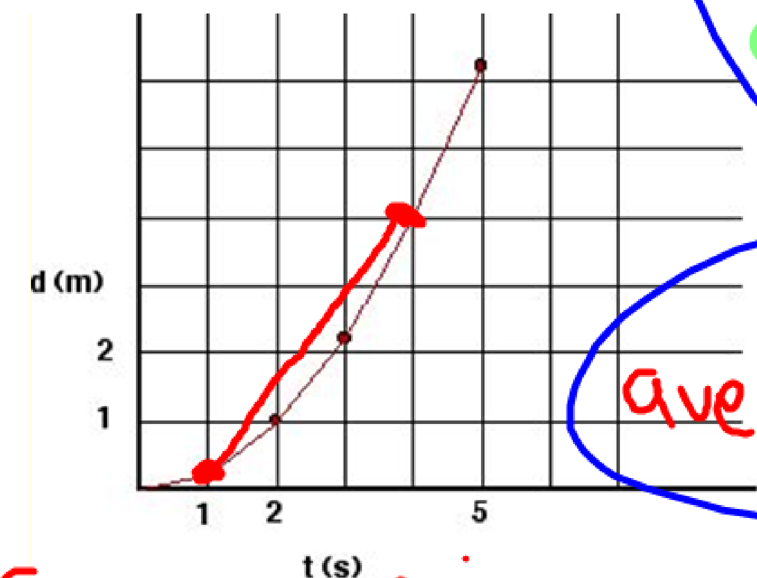
Rate Change $\longrightarrow$ slope

Difference Quotient

Lets analyze the following distance verses time graph.

How would we find the average velocity from t = 1 s to t = 4 s?

$$\text{slope of secant} = \text{ave vel} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$



secant line

$$\text{ave vel} = \frac{f(4) - f(1)}{4 - 1}$$

$$= \frac{4 - \frac{1}{4}}{3} \frac{m}{s}$$

$$= \frac{\frac{15}{4}\,m}{3\,s} = \frac{15}{12} = \frac{5}{4}\,m/s$$

Ex.1 Find the average rate change of $f(x) = x^3 - x$ over the interval [1,3].

$$\text{ave r } \Delta = \frac{f(3) - f(1)}{3 - 1}$$

$$= \frac{\left[(3)^3 - 3\right] - \left[(1)^3 - 1\right]}{2}$$

$$= \frac{24}{2} = 12$$

Application Example

t (years)	2	3	5	7	10
$H(t)$ (meters)	1.5	2	6	11	15

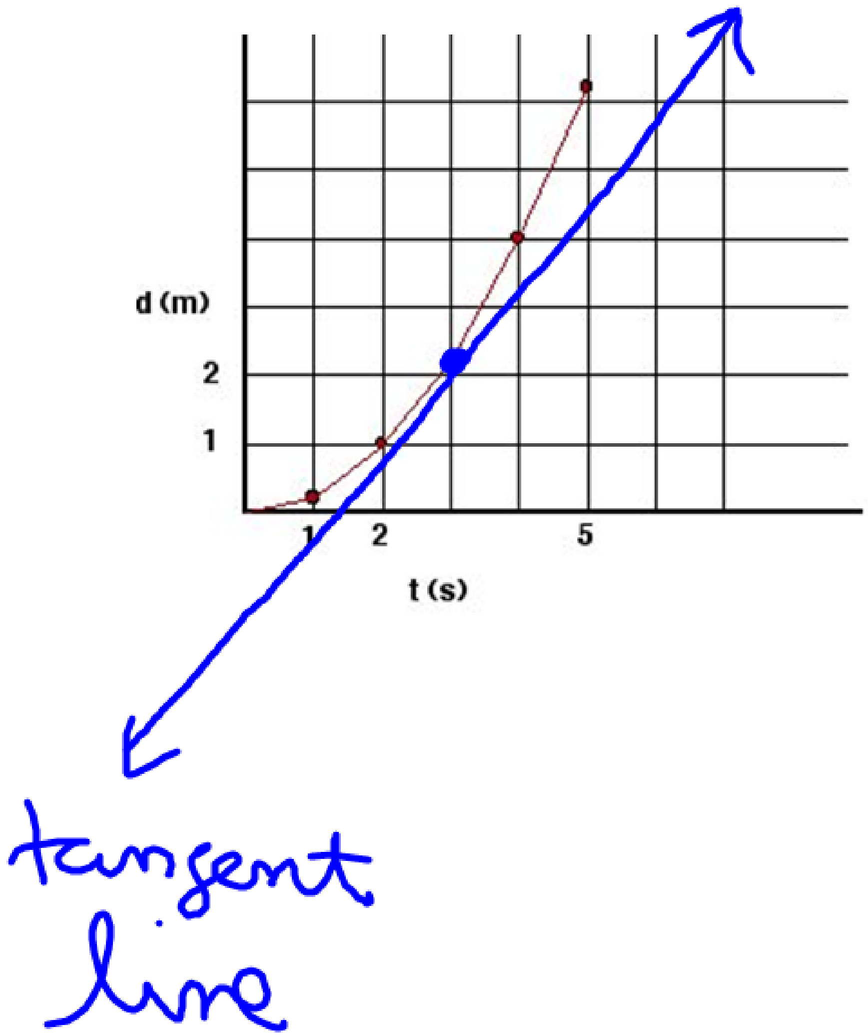
The height of a tree at time t is given by a twice-differentiable function H, where $H(t)$ is measured in meters and t is measured in years. Selected values of $H(t)$ are given in the table above.

Find the average growth rate over the interval [3,10]. Indicate units of measure.

$$\text{ave gr. rate} = \frac{H(10) - H(3)}{10 - 3}$$

$$= \frac{15 - 2}{7}$$

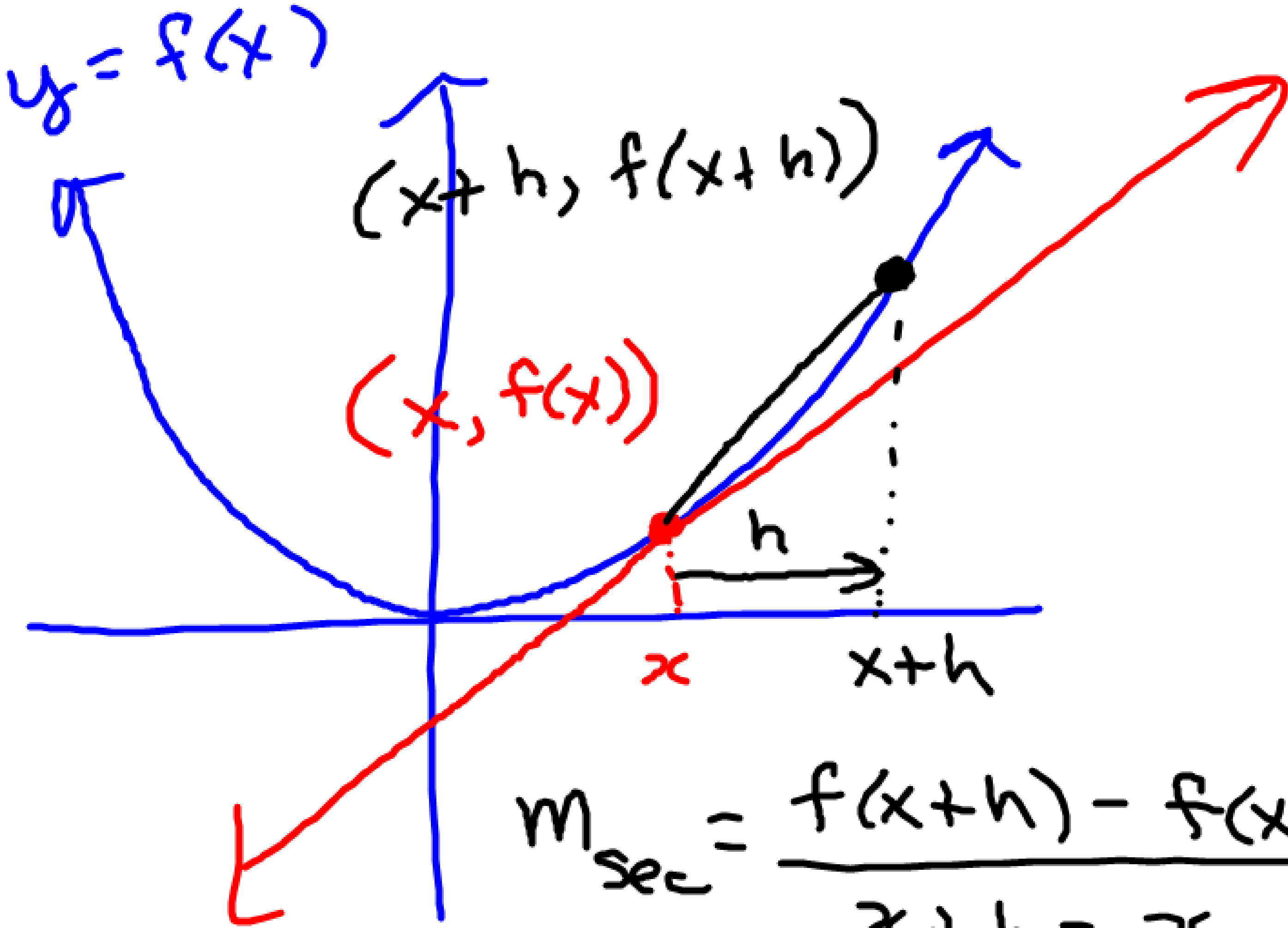
$$= \frac{13}{7} \, \text{m/yr}$$



How would we find the instantaneous velocity at t = 3 s?

Instantaneous rate change is also a slope, but this time it is represented by the **slope of a tangent line!**

Lets develop a formula to help us find the slope of a tangent line!



$$m_{sec} = \frac{f(x+h) - f(x)}{x+h-x}$$

$$m_{sec} = \frac{f(x+h) - f(x)}{h}$$

$$\boxed{m_{tan} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}$$

Slope of a Secant Line Approaching Slop of a Tangent Line

Therefore to find the instantaneous rate change or the slope of a tangent line we have:

$$m_{\tan} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Ex.1 Find the **slope** of the tangent line to the curve $y = 2x^2$ at the point (1,2). Use this slope to find the **equation of the tangent** line at the point (1,2).

$$y - y_1 = m(x - x_1)$$

$$m_{tan} = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{2(x+h)^2 - 2x^2}{h}$$

$$= \lim_{h\to 0} \frac{2(x^2+2xh+h^2) - 2x^2}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{2x^2} + 4xh + 2h^2 - \cancel{2x^2}}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(4x + 2h)}{\cancel{h}}$$

$$= 4x + 2(0)$$

$$\boxed{m_{tan} = 4x}$$

$$m(1) = 4(1)$$

$$m(1) = 4 \qquad (1, 2)$$

$$y - 2 = 4(x - 1)$$

$$y - 2 = 4x - 4$$

$y = 4x - 2$ Slope Intercept (Allowed Fractions)

$y = 4x - 2$ slope intercept

$2 = 4x - y$ Standard Form

No Fractions

x term +

$0 = 4x - y - 2$ General FORM

Example 2 Find the equation of the tangent line to the curve $y = 2x^2 + 4x + 1$ at the point $(2, 17)$.

$4x + 4$

$$m = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h}$$

$$= \lim_{h \to 0} \frac{2(2+h)^2 + 4(2+h) + 1 - 17}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{8} + 8h + 2h^2 \cancel{+8} + 4h \cancel{+1} \cancel{-17}}{h}$$

$$= \lim_{h \to 0} \frac{12h + 2h^2}{h} = \lim_{h \to 0} \frac{\cancel{h}(12 + 2h)}{\cancel{h}} = 12$$

$12 \quad (2, 17)$

$y - y_1 = m(x - x_1)$

$y - 17 = 12(x - 2)$

$y - 17 = 12x - 24$

$\boxed{y = 12x - 7}$

Ex3. Find the **equation of the tangent** line to the curve $y = \frac{1}{x}$ at x = -2.

$(-2, -1/2)$

$$m = \lim_{h \to 0} \frac{f(-2+h) - f(-2)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{1}{-2+h} - \left(-\frac{1}{2}\right)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{2}{2} \cdot \frac{1}{(h-2)} + \frac{1}{2} \cdot \frac{(h-2)}{(h-2)}}{h}$$

$$= \lim_{h \to 0} \frac{\frac{2}{2(h-2)} + \frac{(h-2)}{2(h-2)}}{h}$$

$$= \lim_{h \to 0} \frac{\frac{h}{2(h-2)}}{\frac{h}{1}}$$

$$= \lim_{h \to 0} \frac{\cancel{h}}{2(h-2)} \cdot \frac{1}{\cancel{h}}$$

$$= \frac{1}{2(0-2)} = -1/4$$

$$y+\frac{1}{2}=-\frac{1}{4}(x+2)$$

$$\left(y+\frac{1}{2}=-\frac{1}{4}x-\frac{2}{4}\right)4$$

$$4y+2=-x-2$$

$$x+4y=-4$$

Ex.4 Find the **equation of the tangent** line to the curve $y=\sqrt{x-2}$ at the point (6,2).

$$m=\lim_{h\to 0}\frac{f(6+h)-f(6)}{h}$$

$$=\lim_{h\to 0}\frac{\sqrt{6+h-2}-2}{h}$$

$$=\lim_{h\to 0}\frac{(\sqrt{4+h}-2)}{h}\frac{(\sqrt{4+h}+2)}{(\sqrt{4+h}+2)}$$

$$=\lim_{h\to 0}\frac{\cancel{+h}\,1}{\cancel{h}(\sqrt{4+h}+2)}=\frac{1}{\sqrt{4+0}+2}=\frac{1}{4}$$

$$y-2=\frac{1}{4}(x-6)$$

$$\left(y-2=\frac{1}{4}x-\frac{6}{4}\right)4$$

$$4y-8=x-6$$

$$\boxed{-2=x-4y}$$

Ex.5 Find the point on the function $y = 2x^2 + 7$ where the tangent line is parallel to the line $2y = 8x + 4$.

$y(1) = 2(1)^2 + 7$
$= 9$
$(1, 9)$

$y = 4x + 2$
$m = 4$

$m_{tan} = 4$

$$m = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$4 = \lim_{h \to 0} \frac{2(x+h)^2 + 7 - (2x^2 + 7)}{h}$$

$$4 = \lim_{h \to 0} \frac{\cancel{2x^2} + 4xh + 2h^2 \cancel{+7} - \cancel{2x^2} \cancel{-7}}{h}$$

$$4 = \lim_{h \to 0} \frac{\cancel{h}(4x + 2h)}{\cancel{h}}$$

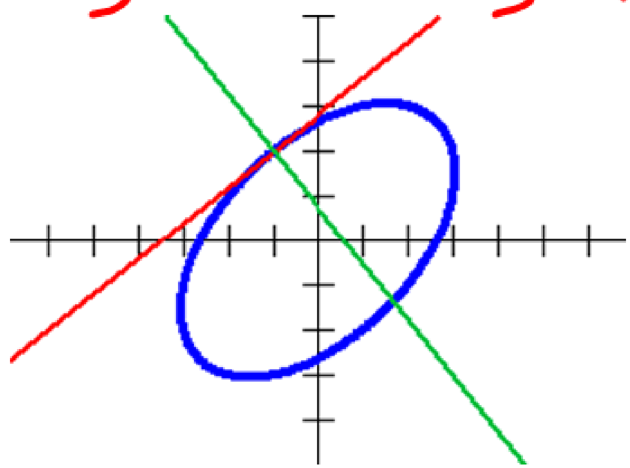
$4 = 4x + 2(0)$

$4 = 4x$
$1 = x$

Normal Line

The **normal line** is a line **perpendicular** to a tangent line.

$m = -\frac{1}{3}$ $3 = m$



Ex. 5 Find the **equation of the normal** to the curve $f(x) = 3 - x^2$ at $x = 1$. $(1, 2)$

$$m = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$$

$$= \lim_{h \to 0} \frac{3 - (1+h)^2 - 2}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{3} - 1 - 2h - h^2 \cancel{- 2}}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(-2-h)}{\cancel{h}} = -2 - 0 = -2$$

$m_{tan} = -2$

$m_{\perp} = \frac{1}{2}$ $(1, 2)$

$y - 2 = \frac{1}{2}(x - 1)$

$y - 2 = \frac{1}{2}x - \frac{1}{2}$

$y = \frac{1}{2}x + \frac{3}{2}$

Assignment
2.4 Handout
AP Textbook Page 87,
#'s 1-4,6,9abc, 10abc, 11abc, 12abc, 29, 30, 31

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2. The graph of the function f, given below, consists of three line segments. Find $\lim_{h \to 0} \dfrac{f(3+h) - f(3)}{h}$.