2.4 Differential Equations

2.2 Differential Equations

Learning Targets:

1. SWBAT solve non-separable differential equations.

2. SWBAT solve separable differential equations.

A **differential equation** is a mathematical equation for an unknown function of one or several variables that relates the values of the function itself and its derivatives of various orders. Differential equations play a prominent role in engineering, physics, economics and other disciplines.

What is a differential equation?

A differential equation contains one or more terms involving derivatives of one variable (the dependent variable, *y*) with respect to another variable (the independent variable, *x*).

For example, $\frac{dy}{dx} = 2x$

$$dy = 2x\,dx$$

$$y = x^2 + C$$

$$\frac{dy}{dx} = \frac{2x}{y}$$

$$\int y\,dy = \int 2x\,dx$$

What does the solutions of a differential equation look like?

Unlike algebraic equations, the solutions of differential equations are functions and not just numbers.

Example 1

Water is leaking from a tank at a rate of $R(t) = 350e^{-0.37t}$ gallons per hour, where t is measured in hours. Find the amount of water that has leaked out after 8 hours.

Non-separable Differential Equation

$$\int_0^8 R(t)\,dt = 896.928 \text{ gallons.}$$

Example 2

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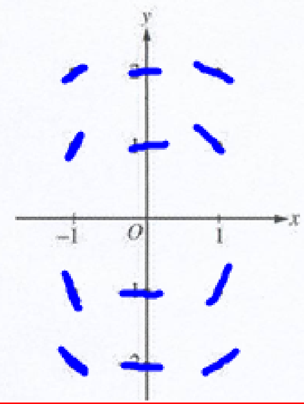
Question 6

Consider the differential equation $\frac{dy}{dx} = -\frac{2x}{y}$.

(a) On the axes provided, sketch a slope field for the given differential equation at the twelve points indicated.
(**Note: Use the axes provided in the pink test booklet.**)

(b) Let $y = f(x)$ be the particular solution to the differential equation with the initial condition $f(1) = -1$. Write an equation for the line tangent to the graph of f at $(1, -1)$ and use it to approximate $f(1.1)$.

(c) Find the particular solution $y = f(x)$ to the given differential equation with the initial condition $f(1) = -1$.



$m = \frac{dy}{dx} = 2 \qquad (1, -1)$

$y + 1 = 2(x - 1)$

$y = 2x - 3$

$y = 2(1.1) - 3$

$= -0.8$

c) $\frac{dy}{dx} = -\frac{2x}{y}$

$\int y\,dy = \int -2x\,dx$

$\boxed{\frac{y^2}{2} = -x^2 + C}$ $\quad \frac{3}{2} = C$

$\frac{(-1)^2}{2} = -(1)^2 + C$

$\frac{1}{2} = -1 + C$

$$\frac{y^2}{2} = -x^2 + \frac{3}{2}$$

$$y^2 = -2x^2 + 3$$

$$y = \pm\sqrt{-2x^2 + 3}$$

$$y = -\sqrt{-2x^2 + 3}$$

Example 3

If $\dfrac{dy}{dx} = 2y^2$ and if $y = -1$ when $x = 1$,

then $y = ?$

Separable Differential Equation

$$\frac{dy}{y^2} = 2\,dx$$

$$\int y^{-2}\,dy = \int 2\,dx$$

$$\frac{y^{-1}}{-1} = 2x + C$$

$$\frac{-1}{y} = 2x + C$$

$$\frac{-1}{-1} = 2(1) + C$$

$$-1 = C$$

$$y\left(\frac{-1}{y}\right)=(2x-1)y$$

$$-1=(2x-1)y$$

$$\frac{-1}{2x-1}=y$$

Example 4

Find the function $f(x)$ if it contains the point $(3,25)$ and has $\frac{dy}{dx} = x\sqrt{y}$.

$$\frac{dy}{\sqrt{y}} = x\,dx$$

$$y^{-1/2}dy = x\,dx$$

$$2y^{1/2} = \frac{x^2}{2} + C$$

$$2(25)^{1/2} = \frac{(3)^2}{2} + C$$

$$10 = \frac{9}{2} + C$$

$$\frac{11}{2} = C$$

$$2\sqrt{y} = \frac{x^2}{2} + \frac{11}{2}$$

$$\sqrt{y} = \frac{x^2}{4} + \frac{11}{4}$$

$$\sqrt{y} = \frac{x^2+11}{4}$$

$$y = \left(\frac{x^2+11}{4}\right)^2$$

$$y = \frac{(x^2+11)^2}{16}$$

Example 5

Find $y = f(x)$ if $\dfrac{dy}{dx} = 2xy$ and if $y = 4$ and $x = 0$.

$x^2 \cdot x^3 = x^{2+3}$

$\int \frac{dy}{y} = \int 2x\,dx$

$\ln|y| = x^2 + C$

$e^{\ln|y|} = e^{x^2+C}$

$|y| = e^{x^2} \cdot e^C$

let $\pm e^C = A$

$y = Ae^{x^2}$

$4 = Ae^{0^2}$

$4 = A$

$y = 4e^{x^2}$

$$\ln|y| = x^2 + C$$

$$\ln 4 = 0 + C$$

$$\ln 4 = C$$

$$e^{\ln|y|} = e^{x^2 + \ln 4}$$

$$|y| = 4e^{x^2} \cdot$$

Example 5 2004B #5

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2004 SCORING GUIDELINES (Form B)

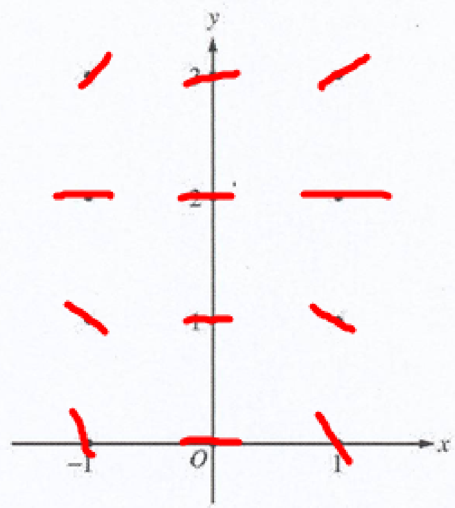
Question 5

Consider the differential equation $\frac{dy}{dx} = x^4(y-2)$.

(a) On the axes provided, sketch a slope field for the given differential equation at the twelve points indicated.
(Note: Use the axes provided in the test booklet.)

(b) While the slope field in part (a) is drawn at only twelve points, it is defined at every point in the xy-plane. Describe all points in the xy-plane for which the slopes are negative.

(c) Find the particular solution $y = f(x)$ to the given differential equation with the initial condition $f(0) = 0$.



$y < 2 \quad x \neq 0$

$$\int \frac{dy}{y-2} = \int x^4\,dx$$

$$\ln|y-2| = \frac{x^5}{5} + C$$

$$e^{\ln|y-2|} = e^{\frac{x^5}{5}+C}$$

$$|y-2| = e^{x^5/5} \cdot e^{C}$$

let $\pm e^{C} = A$

$$y-2 = Ae^{x^5/5}$$

$$0-2 = Ae^{0}$$

$$-2 = A$$

$$y-2 = -2e^{x^5/5}$$

$$y = 2-2e^{x^5/5}$$

Example 6 (2000 #6)

AP Calculus AB–6

Consider the differential equation $\frac{dy}{dx} = \frac{3x^2}{e^{2y}}$.

(a) Find a solution $y = f(x)$ to the differential equation satisfying $f(0) = \frac{1}{2}$.

~~(b) Find the domain and range of the function f found in part (a).~~

$$\int e^{2y}\,dy = \int 3x^2\,dx$$

$$\frac{e^{2y}}{2} = x^3 + C$$

$$\frac{e^{2(1/2)}}{2} = 0^3 + C$$

$$\frac{e}{2} = C$$

$$\frac{e^{2y}}{2} = x^3 + \frac{e}{2}$$

$$e^{2y} = 2x^3 + e$$

$$\ln e^{2y} = \ln(2x^3 + e)$$

$$y = \frac{\ln(2x^3+e)}{2}$$

2013 #6 AP Exam

6. Consider the differential equation $\frac{dy}{dx} = e^y\left(3x^2 - 6x\right)$. Let $y = f(x)$ be the particular solution to the differential equation that passes through $(1, 0)$.

(a) Write an equation for the line tangent to the graph of f at the point $(1, 0)$. Use the tangent line to approximate $f(1.2)$.

(b) Find $y = f(x)$, the particular solution to the differential equation that passes through $(1, 0)$.

a) $\frac{dy}{dx} = e^0\left(3(1)^2 - 6(1)\right) = -3$

$y - 0 = -3(x-1)$

$y = -3x + 3$

$y = -3(1.2) + 3 = -0.6$

$$\frac{dy}{e^y} = (3x^2 - 6x)\,dx$$

$$\int e^{-y}\,dy = \int (3x^2 - 6x)\,dx$$

$$\boxed{-e^{-y} = x^3 - 3x^2 + c}$$

$$-e^{-0} = 1 - 3 + c$$

$$-1 = -2 + c$$

$$1 = c$$

$$-e^{-y} = x^3 - 3x^2 + 1$$

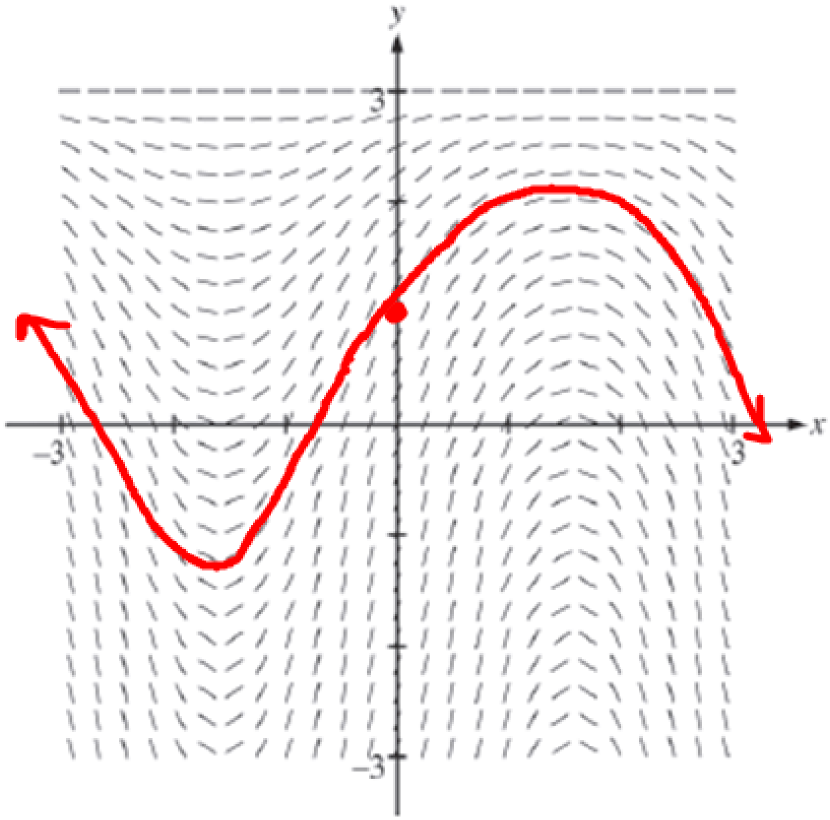
$$e^{-y} = -x^3 + 3x^2 - 1$$

$$\ln e^{-y} = \ln(-x^3 + 3x^2 - 1)$$

$$\Rightarrow y = -\ln(-x^3 + 3x^2 - 1)$$

6. Consider the differential equation $\frac{dy}{dx} = (3 - y)\cos x$. Let $y = f(x)$ be the particular solution to the differential equation with the initial condition $f(0) = 1$. The function f is defined for all real numbers.

(a) A portion of the slope field of the differential equation is given below. Sketch the solution curve through the point $(0, 1)$.



(b) Write an equation for the line tangent to the solution curve in part (a) at the point $(0, 1)$. Use the equation to approximate $f(0.2)$.

(c) Find $y = f(x)$, the particular solution to the differential equation with the initial condition $f(0) = 1$.

b) $\frac{dy}{dx} = (3-1)\cos 0$ $(0,1)$

$= 2$

$y - 1 = 2(x - 0)$

$y = 2x + 1$

$y = 2(0.2) + 1 = 1.4$

c)

$$\frac{dy}{dx} = (3-y)\cos x$$

$$\frac{dy}{dx} = -(y-3)\cos x$$

$$\int \frac{dy}{y-3} = \int -\cos x \, dx$$

$$\ln|y-3| = -\sin x + c$$

$$|y-3| = e^{-\sin x + c}$$

$$|y-3| = e^{-\sin x} \cdot e^{c}$$

S
C
-S
-C
S

let $\pm e^{c} = A$

$$y-3 = Ae^{-\sin x}$$

$$1-3 = Ae^{-\sin 0}$$

$$-2 = A$$

$$y-3 = -2e^{-\sin x}$$

$$y = 3 - 2e^{-\sin x}$$

2012 Free Response

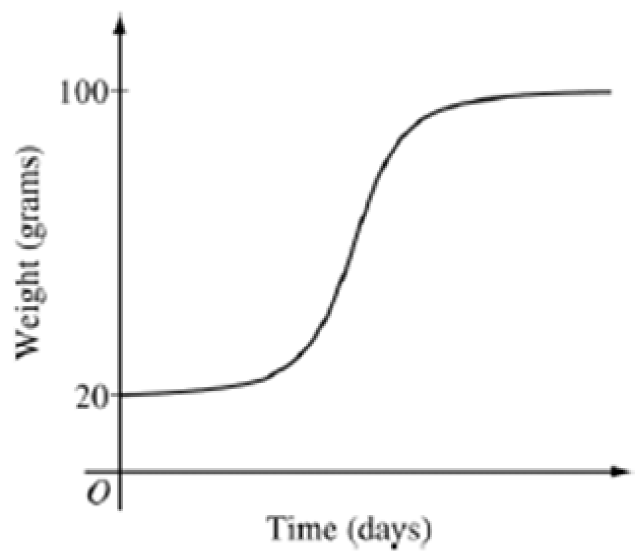
5. The rate at which a baby bird gains weight is proportional to the difference between its adult weight and its current weight. At time $t = 0$, when the bird is first weighed, its weight is 20 grams. If $B(t)$ is the weight of the bird, in grams, at time t days after it is first weighed, then

$$\frac{dB}{dt} = \frac{1}{5}(100 - B).$$

Let $y = B(t)$ be the solution to the differential equation above with initial condition $B(0) = 20$.

(a) Is the bird gaining weight faster when it weighs 40 grams or when it weighs 70 grams? Explain your reasoning.

(b) Find $\frac{d^2B}{dt^2}$ in terms of B. Use $\frac{d^2B}{dt^2}$ to explain why the graph of B cannot resemble the following graph.



(c) Use separation of variables to find $y = B(t)$, the particular solution to the differential equation with initial condition $B(0) = 20$.

c) $\frac{dB}{dt} = \frac{1}{5}(100 - B)$

$\frac{dB}{dt} = -\frac{1}{5}(B - 100)$

$\int \frac{dB}{B-100} = \int -\frac{1}{5}\, dt$

$\ln|B - 100| = -\frac{1}{5}t + C$

$e \qquad\qquad e$

$|B - 100| = e^{-\frac{1}{5}t} \cdot e^{C}$

let $\pm e^{C} = A$

$\boxed{B - 100 = Ae^{-\frac{1}{5}t}}$

$20 - 100 = Ae^{(-\frac{1}{5}\cdot 0)}$

$-80 = A$

$B - 100 = -80e^{-\frac{1}{5}t}$

$B = 100 - 80e^{-\frac{1}{5}t}$

a) $\frac{dB}{dt} = \frac{1}{5}(100 - 40) = 12$

$\frac{dB}{dt} = \frac{1}{5}(100 - 70) = 6$

b) $\frac{dB}{dt} = \frac{1}{5}(100 - B)$

$$\frac{d^2B}{dt^2} = \frac{1}{5}\left(-1\,\frac{dB}{dt}\right)$$

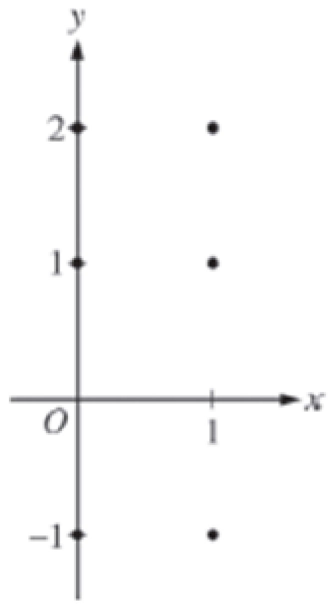
$$= -\frac{1}{5}\,{}^{dB}/_{dt}$$

$$= -\frac{1}{5}\left(\frac{1}{5}(100 - B)\right)$$

$$= -\frac{1}{25}(100 - B) < 0 \quad \therefore \text{ graph } B \quad (D)$$

4. Consider the differential equation $\frac{dy}{dx} = 2x - y$.

(a) On the axes provided, sketch a slope field for the given differential equation at the six points indicated.



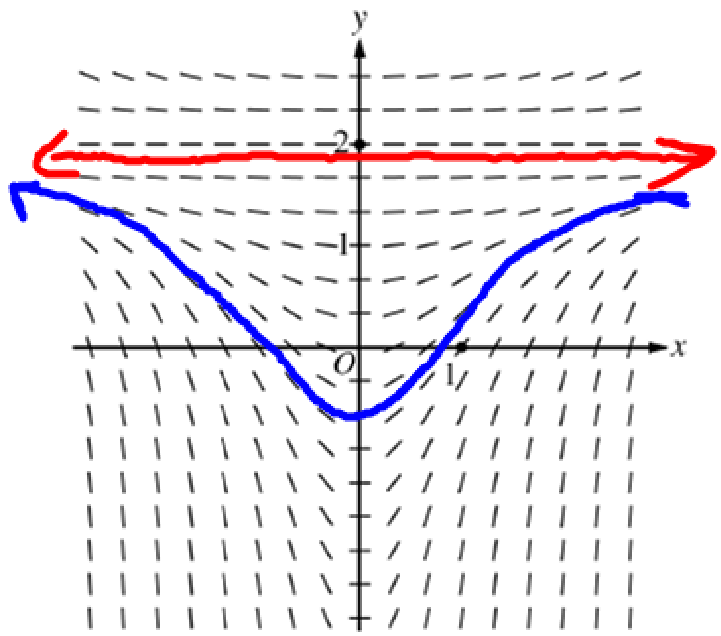
(b) Find $\frac{d^2y}{dx^2}$ in terms of x and y. Determine the concavity of all solution curves for the given differential equation in Quadrant II. Give a reason for your answer.

(c) Let $y = f(x)$ be the particular solution to the differential equation with the initial condition $f(2) = 3$. Does f have a relative minimum, a relative maximum, or neither at $x = 2$? Justify your answer.

(d) Find the values of the constants m and b for which $y = mx + b$ is a solution to the differential equation.

6. Consider the differential equation $\frac{dy}{dx} = \frac{1}{3}x(y-2)^2$.

(a) A slope field for the given differential equation is shown below. Sketch the solution curve that passes through the point $(0, 2)$, and sketch the solution curve that passes through the point $(1, 0)$.



(b) Let $y = f(x)$ be the particular solution to the given differential equation with initial condition $f(1) = 0$. Write an equation for the line tangent to the graph of $y = f(x)$ at $x = 1$. Use your equation to approximate $f(0.7)$.

(c) Find the particular solution $y = f(x)$ to the given differential equation with initial condition $f(1) = 0$.

b)

$$\frac{dy}{dx} = \frac{1}{3}(1)(0-2)^2$$

$$= \frac{1}{3}(4)$$

$$= 4/3$$

$$y - 0 = \frac{4}{3}(x-1)$$

$$y = \frac{4}{3}(0.7-1) = \frac{4}{3}\left(-\frac{3}{10}\right) = -2/5$$

c) $\dfrac{dy}{(y-2)^2} = \dfrac{1}{3}x\,dx$

$(y-2)^{-2}\,dy = \dfrac{1}{3}x\,dx$

$\dfrac{(y-2)^{-1}}{-1} = \dfrac{1}{3\cdot 2}x^2 + C$

$$\boxed{\dfrac{-1}{Y-2} = \dfrac{1}{6}x^2 + C}$$

$\dfrac{-1}{-2} = \dfrac{1}{6}(1)^2 + C$

$\dfrac{1}{2} = \dfrac{1}{6} + C$

$\dfrac{1}{3} = C$

$$\frac{-1}{y-2} = \frac{1}{6}x^2 + \frac{2}{6}$$

$$\frac{-1}{y-2} = \frac{x^2+2}{6}$$

$$-6 = (y-2)(x^2+2)$$

$$\frac{-6}{x^2+2} = y-2$$

$$2 - \frac{6}{x^2+2} = y$$

The graph $y = f(x)$ is defined for all $x \geq 0$, and contains the point (0,1). If $\frac{dy}{dx} = 3\sqrt{xy}$ and $f(x) > 0$, for all x, then find the particular solution $f(x)$.

$\sqrt{2} \cdot \sqrt{3} = \sqrt{6}$

$$\frac{dy}{dx} = 3\sqrt{xy}$$

$$\frac{dy}{dx} = 3\sqrt{x} \cdot \sqrt{y}$$

$$\frac{dy}{\sqrt{y}} = 3\sqrt{x}\, dx$$

$$\int y^{-1/2}\,dy = \int 3x^{1/2}\,dx$$

$$\boxed{2y^{1/2} = 2x^{3/2} + C}$$

$$2\sqrt{1} = 2(0) + C$$

$$2 = C$$

$$2\sqrt{y} = 2x^{3/2} + 2$$

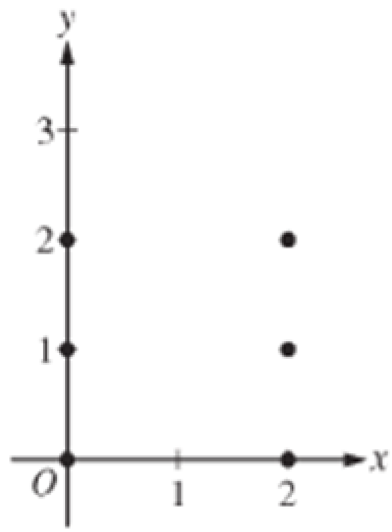
$$\sqrt{y} = x^{3/2} + 1$$

$$y = (x^{3/2} + 1)^2$$

Assignment Handout

4. Consider the differential equation $\frac{dy}{dx} = \frac{y^2}{x - 1}$.

(a) On the axes provided, sketch a slope field for the given differential equation at the six points indicated.



(b) Let $y = f(x)$ be the particular solution to the given differential equation with the initial condition $f(2) = 3$. Write an equation for the line tangent to the graph of $y = f(x)$ at $x = 2$. Use your equation to approximate $f(2.1)$.

(c) Find the particular solution $y = f(x)$ to the given differential equation with the initial condition $f(2) = 3$.

5. At the beginning of 2010, a landfill contained 1400 tons of solid waste. The increasing function W models the total amount of solid waste stored at the landfill. Planners estimate that W will satisfy the differential equation $\frac{dW}{dt} = \frac{1}{25}(W - 300)$ for the next 20 years. W is measured in tons, and t is measured in years from the start of 2010.

   (a) Use the line tangent to the graph of W at $t = 0$ to approximate the amount of solid waste that the landfill contains at the end of the first 3 months of 2010 (time $t = \frac{1}{4}$).

   (b) Find $\frac{d^2W}{dt^2}$ in terms of W. Use $\frac{d^2W}{dt^2}$ to determine whether your answer in part (a) is an underestimate or an overestimate of the amount of solid waste that the landfill contains at time $t = \frac{1}{4}$.

   (c) Find the particular solution $W = W(t)$ to the differential equation $\frac{dW}{dt} = \frac{1}{25}(W - 300)$ with initial condition $W(0) = 1400$.

2010

6. Solutions to the differential equation $\frac{dy}{dx} = xy^3$ also satisfy $\frac{d^2y}{dx^2} = y^3\left(1 + 3x^2y^2\right)$. Let $y = f(x)$ be a particular solution to the differential equation $\frac{dy}{dx} = xy^3$ with $f(1) = 2$.

   (a) Write an equation for the line tangent to the graph of $y = f(x)$ at $x = 1$.

   (b) Use the tangent line equation from part (a) to approximate $f(1.1)$. Given that $f(x) > 0$ for $1 < x < 1.1$, is the approximation for $f(1.1)$ greater than or less than $f(1.1)$? Explain your reasoning.

   (c) Find the particular solution $y = f(x)$ with initial condition $f(1) = 2$.

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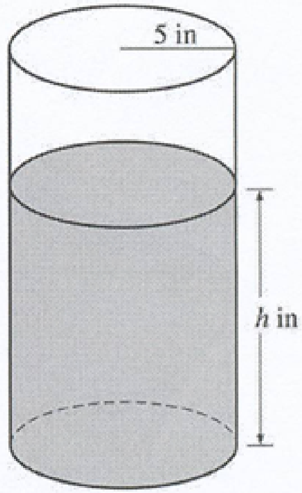
Question 5

A coffeepot has the shape of a cylinder with radius 5 inches, as shown in the figure above. Let h be the depth of the coffee in the pot, measured in inches, where h is a function of time t, measured in seconds. The volume V of coffee in the pot is changing at the rate of $-5\pi\sqrt{h}$ cubic inches per second. (The volume V of a cylinder with radius r and height h is $V = \pi r^2 h$.)



(a) Show that $\dfrac{dh}{dt} = -\dfrac{\sqrt{h}}{5}$.

(b) Given that $h = 17$ at time $t = 0$, solve the differential equation $\dfrac{dh}{dt} = -\dfrac{\sqrt{h}}{5}$ for h as a function of t.

(c) At what time t is the coffeepot empty?

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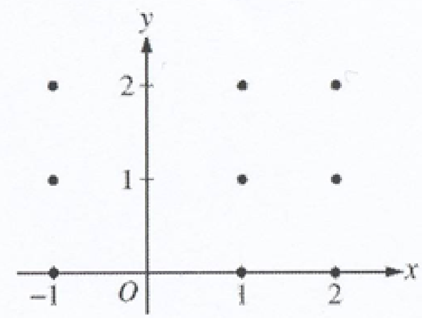
Question 5

Consider the differential equation $\frac{dy}{dx} = \frac{y-1}{x^2}$, where $x \neq 0$.

(a) On the axes provided, sketch a slope field for the given differential equation at the nine points indicated.
(Note: Use the axes provided in the exam booklet.)



(b) Find the particular solution $y = f(x)$ to the differential equation with the initial condition $f(2) = 0$.

(c) For the particular solution $y = f(x)$ described in part (b), find $\lim_{x \to \infty} f(x)$.

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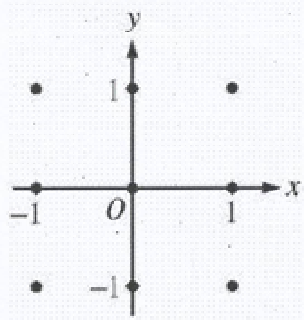
2006 SCORING GUIDELINES (Form B)

Question 5

Consider the differential equation $\frac{dy}{dx} = (y-1)^2 \cos(\pi x)$.

(a) On the axes provided, sketch a slope field for the given differential equation at the nine points indicated. **(Note: Use the axes provided in the exam booklet.)**



(b) There is a horizontal line with equation $y = c$ that satisfies this differential equation. Find the value of c.

(c) Find the particular solution $y = f(x)$ to the differential equation with the initial condition $f(1) = 0$.