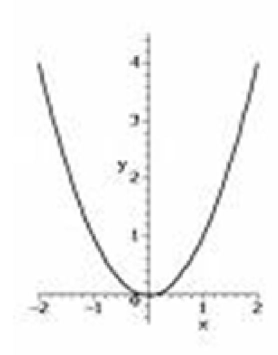
2.3 Continuity

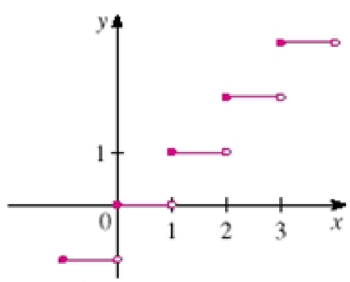
Defn: **Continuous functions** are ones whose graphs can be drawn without lifting your pencil off the paper.

Example $y = x^2$



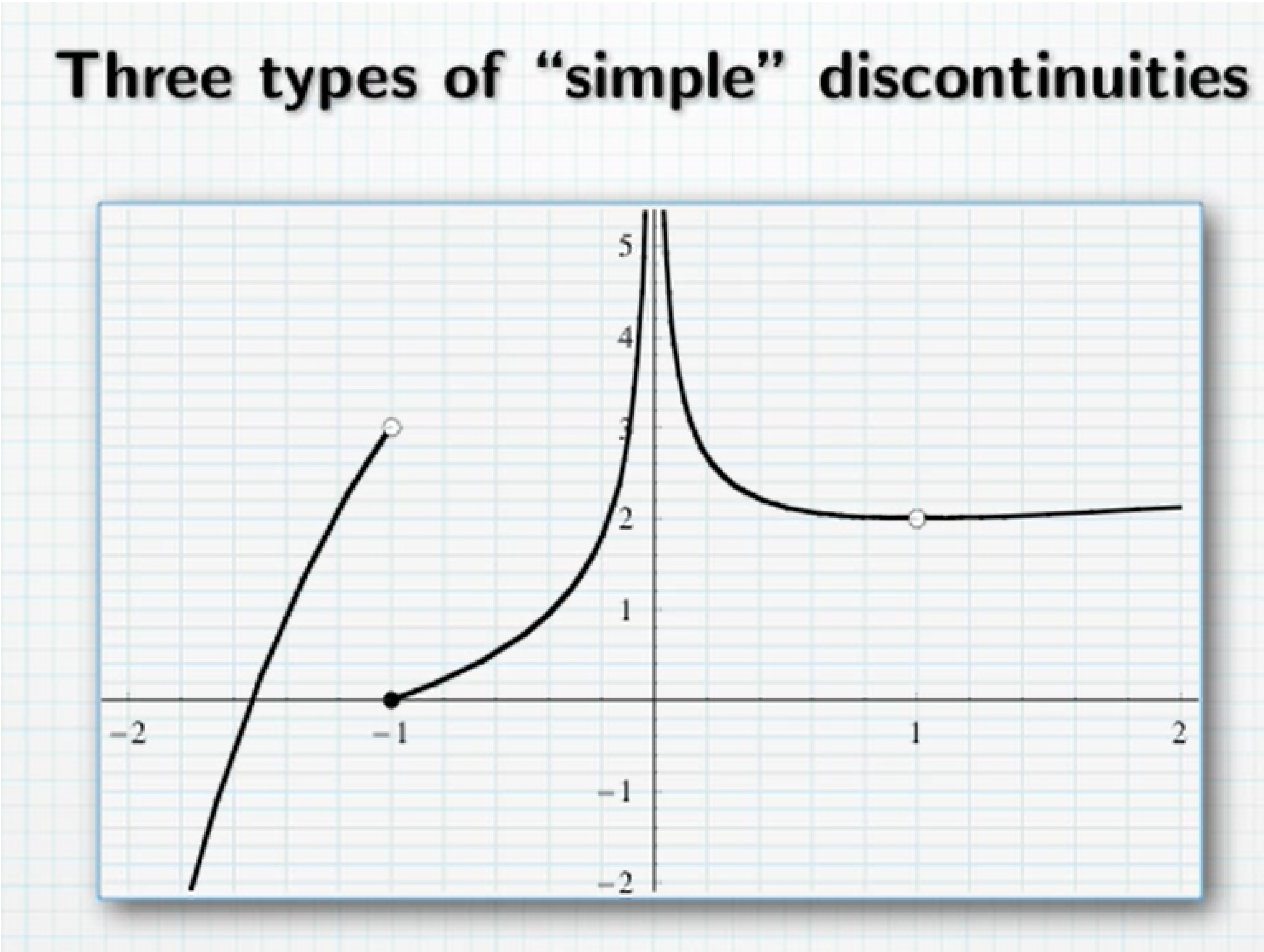
Non-example $y = [x]$

Greatest Integer Function



(d) $f(x) = [\![x]\!]$

Types of Discontinuities



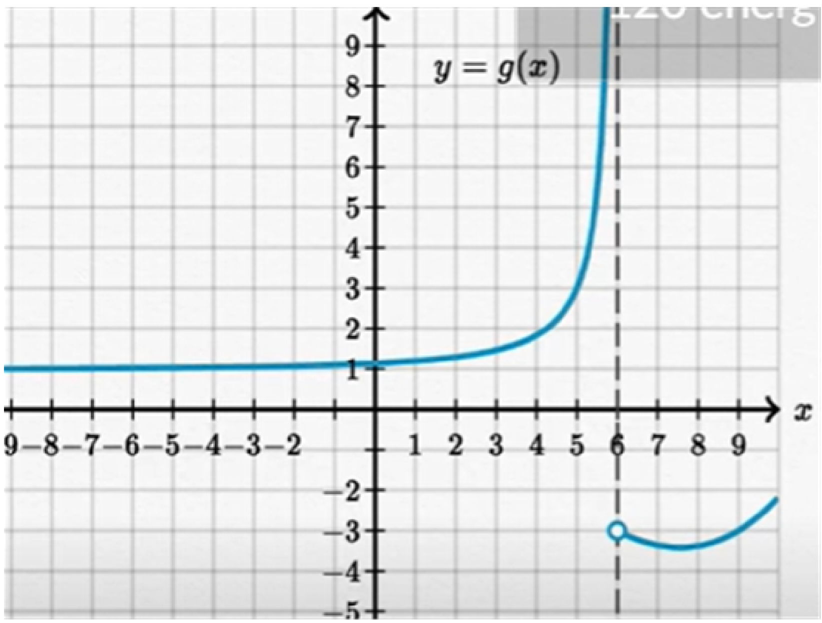
Continuity at a Point

Mathematically, a function is continuous at a number "*a*" if:

i) $f(a)$ is defined

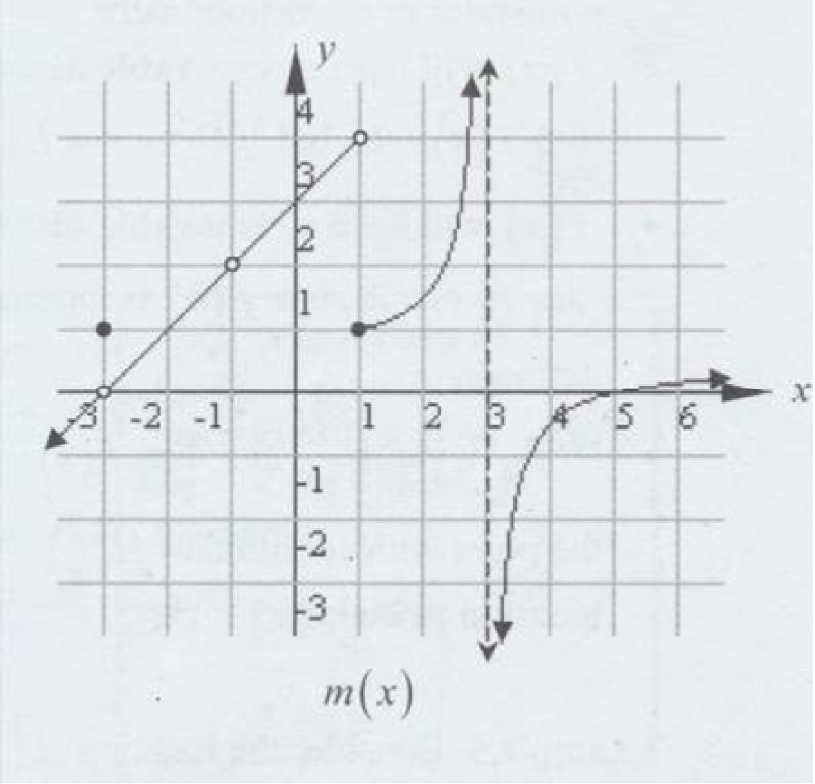
ii) $\lim_{x \to a} f(x)$ exists

iii) $\lim_{x \to a} f(x) = f(a)$

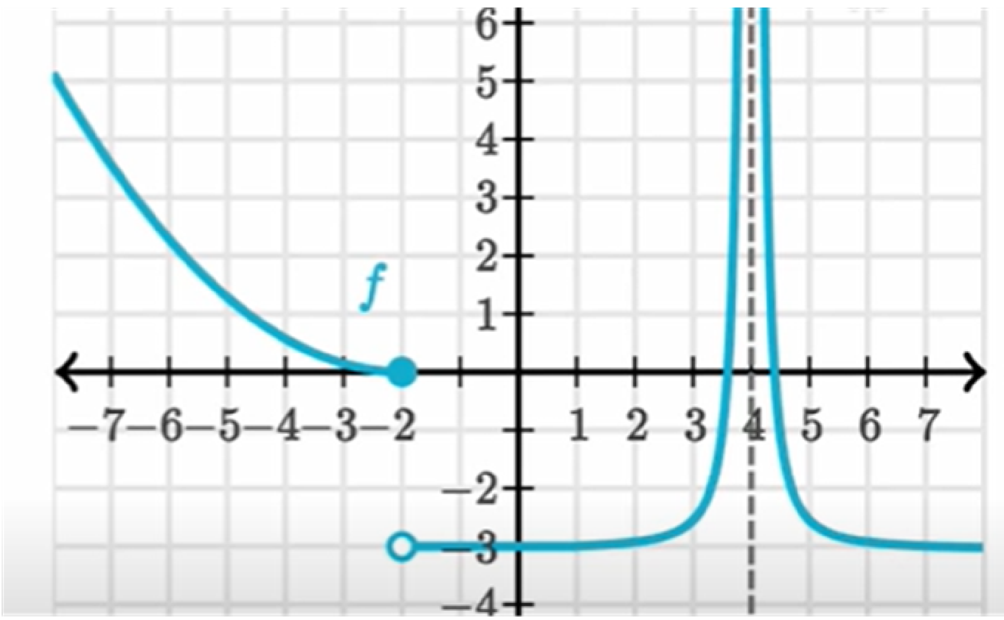


- Both $\lim_{x\to6^+} g(x)$ and $\lim_{x\to6^-} g(x)$ exist
- $\lim_{x\to6} g(x)$ exists
- g is defined at $x = 6$
- g is continuous at $x = 6$
- None of the above

Ex.1 By examining the following graph of *m(x)* explain why *m(x)* is discontinuous at the following points: a) x = 3 b) x = 1 c) x = -1 d) x = -3. Classify each discontinuity.



Continuity Over an Interval



AP® CALCULUS AB
2003 SCORING GUIDELINES

Question 6

Let f be the function defined by

$$f(x) = \begin{cases} \sqrt{x+1} & \text{for } 0 \le x \le 3 \\ 5 - x & \text{for } 3 < x \le 5. \end{cases}$$

(a) Is f continuous at $x = 3$? Explain why or why not.

2011 AP Exam #6

6. Let f be a function defined by $f(x) = \begin{cases} 1 - 2\sin x & \text{for } x \le 0 \\ e^{-4x} & \text{for } x > 0. \end{cases}$

(a) Show that f is continuous at $x = 0$.

(b) For $x \neq 0$, express $f'(x)$ as a piecewise-defined function. Find the value of x for which $f'(x) = -3$.

(c) Find the average value of f on the interval $[-1, 1]$.

Functions That Are Continuous

Continuity Principles

1. All constant functions are continuous.
2. The following types of functions are continuous at every point in their domain: polynomial, rational, power, root, trigonometric, exponential, and logarithmic.
3. If $f(x)$ and $g(x)$ are continuous functions, so are $(f+g)(x)$, $(f-g)(x)$, $(fg)(x)$, and $\left(\frac{f}{g}\right)(x)$ in their common domains provided, in the last case, that $g(x) \neq 0$.

Making Functions Continuous

Ex. Find the value of "a" so that our function is continuous.

$$a)\ \ f(x)=\begin{cases} x^2, x<1 \\ 2ax-1, x\geq 1 \end{cases}$$

$$f(x)=\begin{cases} x^2-3, x\le -1 \\ cx-d, -1<x<3 \\ x^2+1, x\ge 3 \end{cases}$$

The function $f(x) = \dfrac{6x^2 + 18x + 12}{x^2 - 4}$ is not defined at $x = \pm 2$. What value should be assigned to $f(-2)$ to make $f(x)$ continuous at that point?

Let f be the function given by

$$f(x) = \begin{cases} \dfrac{\sqrt{x+4}-3}{x-5} & \text{if } x \neq 5 \\ c & \text{if } x = 5\,. \end{cases}$$

If f is continuous at $x = 5$, then what is the value of c?

$$f(x) = \begin{cases} \dfrac{x^2 - 7x + 10}{b(x-2)} & \text{for } x \neq 2 \\ b & \text{for } x = 2 \end{cases}$$

10. Let f be the function defined above. For what value of b is f continuous at $x = 2$?

(A) -3 (B) $\sqrt{2}$ (C) 3 (D) 5 (E) There is no such value of b.

Finding and Classifying Points of Discontinuity

Ex.4 Find and classify the discontinuities, if any, for each of the following:

a) $f(x)=\dfrac{x^3+1}{x+1}$

b) $f(x)=\dfrac{x^2+4x}{x^3-4x}$

c) $f(x)=\left\{\begin{array}{l} x^2, x \leq 2 \\ 2x+1, x>2 \end{array}\right\}$

d) $f(x)=\begin{cases} x^2+4, \text{ if } x \le 0 \\ x+1, \text{ if } 0 < x \le 1 \\ x^2+1, \text{ if } x > 1 \end{cases}$

e) $f(x) = \frac{|x^2 - x - 20|}{x - 5}$

6. Functions f, g, and h are twice-differentiable functions with $g(2) = h(2) = 4$. The line $y = 4 + \frac{2}{3}(x - 2)$ is tangent to both the graph of g at $x = 2$ and the graph of h at $x = 2$.

(a) Find $h'(2)$.

(b) Let a be the function given by $a(x) = 3x^3h(x)$. Write an expression for $a'(x)$. Find $a'(2)$.

(c) The function h satisfies $h(x) = \frac{x^2 - 4}{1 - (f(x))^3}$ for $x \neq 2$. It is known that $\lim_{x \to 2} h(x)$ can be evaluated using L'Hospital's Rule. Use $\lim_{x \to 2} h(x)$ to find $f(2)$ and $f'(2)$. Show the work that leads to your answers.

(d) It is known that $g(x) \leq h(x)$ for $1 < x < 3$. Let k be a function satisfying $g(x) \leq k(x) \leq h(x)$ for $1 < x < 3$. Is k continuous at $x = 2$? Justify your answer.

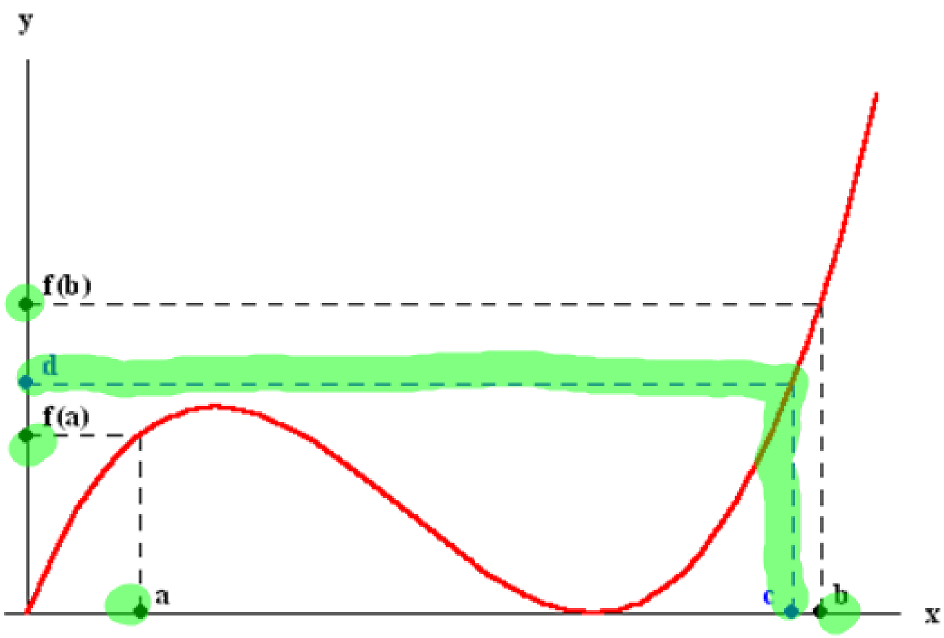
Assignment
Page 80 AP Text
#'s
11-16,19,20,23,24,35-38,41
Page 147 Calc 30 Text
#'s 1,4-7,10b,c,e,k,l,m

Intermediate Value Theorem For Continuous Functions

Introduction

Intermediate Value Theorem

Let $f(x)$ be a continuous function on the interval $[a, b]$. If $d \in [f(a), f(b)]$, then there is a $c \in [a, b]$ such that $f(c) = d$.



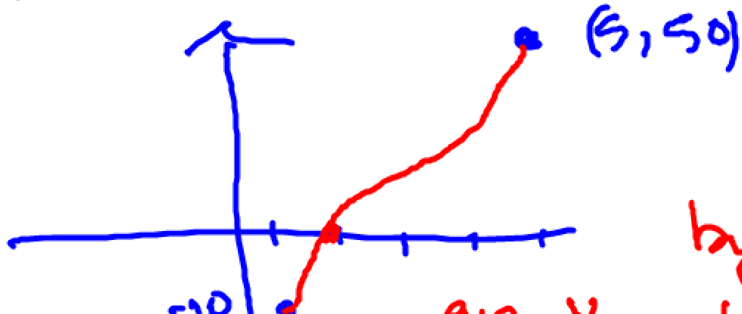
Example

Let h be the function defined by $h(x) = x^3 - 2x^2 - 4x - 5$. Can the Intermediate Value Theorem be applied to guarantee a value in the interval $[1, 5]$ where $h(x) = 0$? Justify your answer.

Since $h(x)$ polynomial ∴ it is continuous and we have closed interval $[1, 5]$ ∴ IVT can apply.

$h(1) = (1)^3 - 2(1)^2 - 4(1) - 5 = -10$

$h(5) = (5)^3 - 2(5)^2 - 4(5) - 5 = 50$



Since $h(1) = -10 < 0 < h(5) = 50$ and since h cont. by IVT there exists an x, $1 < x < 5$ such that $h(x) = 0$

Examples.

x	20	22	24	26	28	30
$f(x)$	2	1	-3	-2	-3	1

Selected values of a continuous function f are given in the table above. What is the fewest number of zeros of f in the interval $[20, 30]$?

(A) Zero, because $f(x)$ is never shown to be zero.

(B) One, because $f(x)$ changes from negative to positive between $x = 28$ and $x = 30$.

(C) Two, because the $f(x)$ changes sign when $22 < x < 24$ and when $28 < x < 30$.

(D) Three, because $f(x)$ starts positive, is then negative, and then positive.

Examples.

x	4	5	6	7	8	9
$k(x)$	2	5	1	-1	-1	2

Values of the function $k(x)$ are given in the table above. Which of the following is enough alone to guarantee a c in $[4, 9]$ such that $k(c) = 4$?

I k is continuous on $[5, 6]$.

II k is continuous on $[4, 5]$.

III k is defined for every value on $[4, 9]$.

(A) I and II only (B) III only (C) I, II, and III (D) None of these

The **Intermediate Value Theorem** is usually used for justification:

continuous

AP® CALCULUS AB
2007 SCORING GUIDELINES

Question 3

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
1	6	4	2	5
2	9	2	3	1
3	10	-4	4	2
4	-1	3	6	7

The functions f and g are differentiable for all real numbers, and g is strictly increasing. The table above gives values of the functions and their first derivatives at selected values of x. The function h is given by $h(x) = f(g(x)) - 6$.

(a) Explain why there must be a value r for $1 < r < 3$ such that $h(r) = -5$.

cont

$h(x) = f(g(x)) - 6$

$h(1) = f(g(1)) = f(2) = 9$

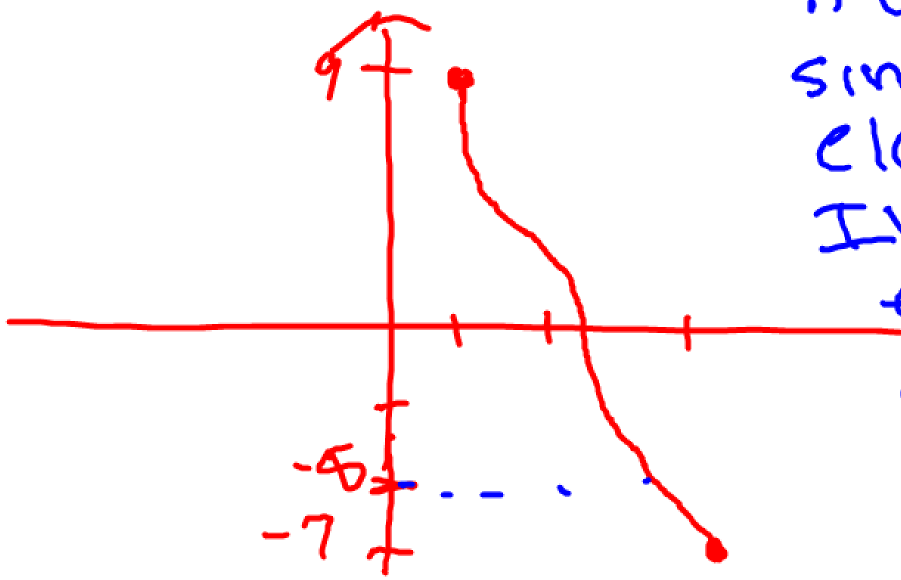
$h(1) = 9$

$h(3) = f(g(3)) - 6$

$= f(4) - 6$

$= -1 - 6 = -7$

$h(3) = -7$



$h(1) = 9 > -5 > h(3) = -7$

since h cont and closed [1,3] then IVT applies. There exists r

$1 < r < 3$ such that

$h(r) = -5$