2.2 Properties of Limits and Algebraically Evaluating Limits.

Strategies to Evaluate Limits Algebraically

1. Evaluating Limits By Direct Substitution

Ex.1 Find $\lim_{x\to 5}\left(x^2+2x-3\right)$

$$\lim_{x\to 5} x^2 + \lim_{x\to 5} 2x - \lim_{x\to 5} 3$$

$$= (5)^2 + 2(5) - 3$$

$$= 25+10-3$$

$$= 32$$

$$(5)^2+2(5)-3$$

$$= 32$$

Ex.2 Find $\lim_{x\to 1} \frac{x^4 - 5x^2 + 1}{x+2}$

$$= \frac{(1)^4 - 5(1)^2 + 1}{1+2}$$

$$= \frac{-3}{3}$$

$$= -1$$

Ex.3 Find $\lim_{x \to 3} \sqrt{x^2 + x}$

$$= \sqrt{(3)^2 + 3}$$

$$= \sqrt{12}$$

$$= 2\sqrt{3}$$

Example: Find $\lim\limits_{x \to 0} \dfrac{\sin x + 2\cos x + \cos^2 x}{\sin x + \cos 2x}$.

$$\frac{\cancel{\sin 0} + 2\cos 0 + (\cos 0)^2}{\cancel{\sin 0} + \cos(2 \cdot 0)}$$

$$\frac{2(1) + (1)^2}{1} = 3$$

$$\lim_{x\to 9} \frac{\log_3 x}{\sin\left(\frac{\pi x}{18}\right)}$$

$$= \lim_{x\to 9} \frac{\log_3 9}{\sin \frac{9\pi}{18}}$$

$$= \lim_{x\to 9} \frac{2}{\cancel{\sin\frac{\pi}{2}}\, 1} = 2$$

If our function is a **polynomial** we can use **direct substitution** to evaluate our limit. If our function is **rational or algebraic**, we can use direct substitution provided it does not make our function undefined.

2. Evaluating One Sided Limits

Ex.1 Determine each of the following limits:

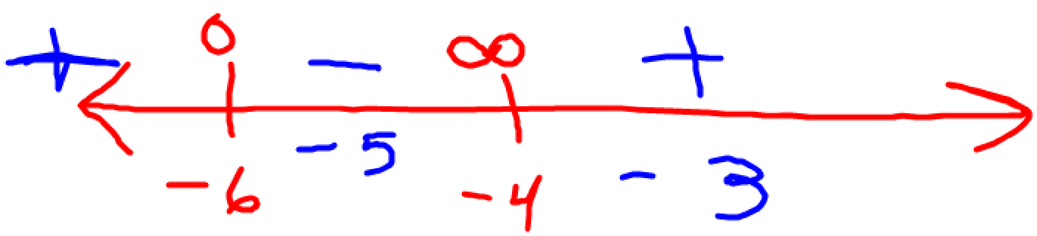
a) $\lim_{x\to -4^{+}} \frac{x+6}{x+4}$ ∞

b) $\lim_{x\to -4^{-}} \frac{x+6}{x+4}$ $-\infty$

c) $\lim_{x\to -4} \frac{x+6}{x+4}$ DNE

$$\lim_{x \to -4} \frac{x+6}{x+4} = \frac{2}{0} \quad \text{UND}$$

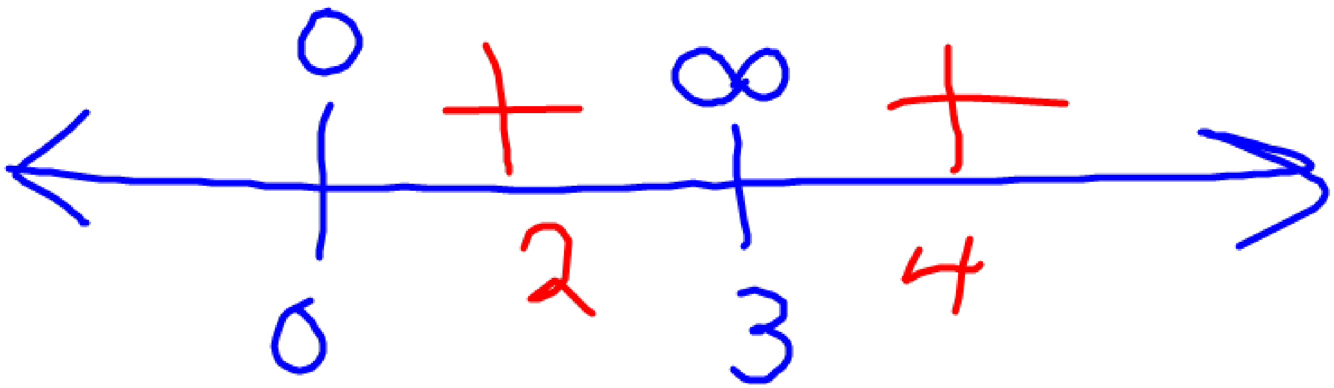
$x = -4$ VA



d) $\lim_{x\to 3^-} \frac{x}{(x-3)^2}$ $\frac{\infty}{+}$

$\frac{3}{0}$

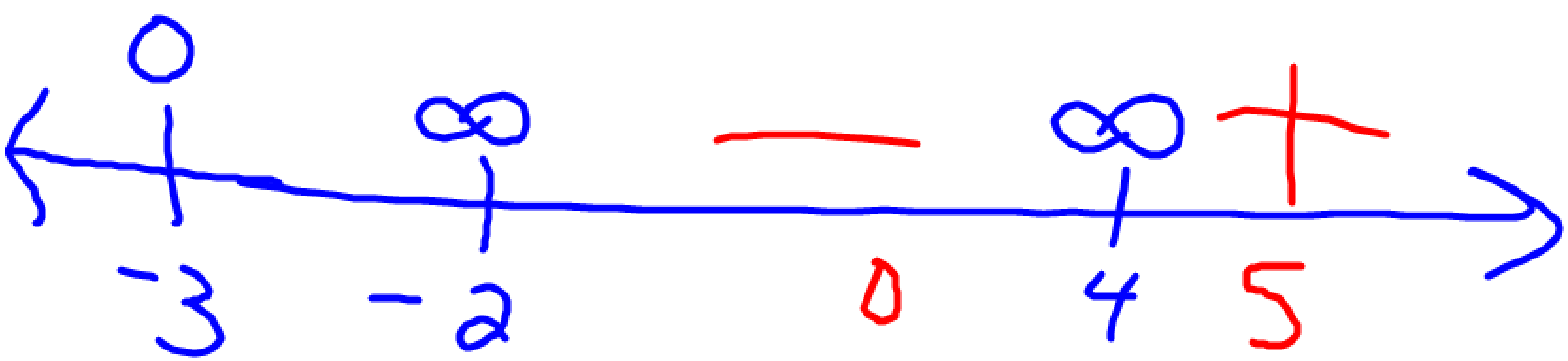
$x = 3$ VA



$$\lim_{x\to 4} \frac{x+3}{x^2-2x-8} \qquad \frac{7}{0} \qquad x=4 \text{ VA}$$

DNE

$$\lim_{x\to 4} \frac{(x+3)}{(x-4)(x+2)}$$



Ex.2 Let $f(x)=\begin{cases} x^2, \text{ if } x \le 1 \\ 2-x, \text{ if } x > 1 \end{cases}$ find the limits if they exist.

$a) \lim_{x \to -3} f(x)$

$\lim_{x \to -3} x^2 = (-3)^2$
$= 9$

$b) \lim_{x \to 5} f(x)$

$= \lim_{x \to 5} 2-x$

$= 2-5 = -3$

$c) \lim_{x \to 1} f(x) = 1$

$\lim_{x \to 1^-} x^2 = 1$

$\lim_{x \to 1^+} 2-x = 1$

Assignment
Page 138 (Calc 30 Text)
#'s 1-7, 10 - 13, 16 - 18, 61 - 63

3. Evaluating Limits By Factoring First

Ex.3 Determine each of the following limits:

a) $\lim\limits_{x \to 6} \dfrac{x^2 - 7x + 6}{x^2 - 36}$

$\lim\limits_{x \to 6} \dfrac{\cancel{(x-6)}(x-1)}{\cancel{(x-6)}(x+6)}$

$= \lim\limits_{x \to 6} \dfrac{(x-1)}{(x+6)}$

$= \dfrac{6-1}{6+6} = 5/12$

b) $\lim\limits_{x \to 2} \dfrac{x^3 - 8}{x^2 - 3x + 2}$

$\lim\limits_{x \to 2} \dfrac{\cancel{(x-2)}(x^2+2x+4)}{\cancel{(x-2)}(x-1)}$

$= \dfrac{(2)^2 + 2(2) + 4}{2-1}$

$= 12$

4. Evaluating Limits By Simplifying

Ex.4 Determine each of the following limits:

a) $\lim_{h\to 0}\frac{(2+h)^2-4}{h}$

$$= \lim_{h\to 0}\frac{\cancel{4}+4h+h^2\cancel{-4}}{h}$$

$$= \lim_{h\to 0}\frac{\cancel{h}(4+h)}{\cancel{h}} = 4+0 = 4$$

$$\text{b) } \lim_{x\to 0} \frac{\left(\frac{3}{3}\right)\frac{1}{x+3} - \frac{1}{3}\left(\frac{x+3}{x+3}\right)}{2x}$$

$$= \lim_{x\to 0} \frac{\frac{3}{3(x+3)} - \frac{(x+3)}{3(x+3)}}{2x}$$

$$= \lim_{x\to 0} \frac{\left(\frac{3-x-3}{3(x+3)}\right)}{2x}$$

$$= \lim_{x\to 0} \frac{-\cancel{x}}{3(x+3)} \cdot \frac{1}{2\cancel{x}} = \lim_{x\to 0} \frac{-1}{6(x+3)}$$

$$= \frac{-1}{6(0+3)} = -\frac{1}{18}$$

5. Evaluating Limits By Rationalizing

a) $\lim_{x \to 0} \frac{\sqrt{x+1}-1}{x}$ $\frac{(\sqrt{x+1}+1)}{(\sqrt{x+1}+1)}$

$$= \lim_{x \to 0} \frac{x+1-1}{x(\sqrt{x+1}+1)}$$

$$= \lim_{x \to 0} \frac{\cancel{x}}{\cancel{x}(\sqrt{x+1}+1)}$$

$$= \frac{1}{\sqrt{0+1}+1} = \frac{1}{2}$$

$$\text{b) } \lim_{x\to 6} \frac{\sqrt{3+r}-3}{r-6} \cdot \frac{(\sqrt{3+r}+3)}{(\sqrt{3+r}+3)}$$

$$= \lim_{x\to 6} \frac{3+r-9}{(r-6)(\sqrt{3+r}+3)}$$

$$= \lim_{x\to 6} \frac{\cancel{(r-6)}\,1}{\cancel{(r-6)}(\sqrt{3+r}+3)}$$

$$= \frac{1}{\sqrt{3+6}+3} = \frac{1}{6}$$

Assignment Page 139
#'s 19 - 26, 28, 29, 31 - 38

6. Evaluating Limits At Infinity Of Rational Functions

Evaluate the following limits:

a) $\lim\limits_{x\to\infty} \dfrac{6x-5}{2x+3}$

$$\frac{\frac{1}{x}}{\frac{1}{x}} = \lim_{x\to\infty} \frac{6-\frac{5}{x}}{2+\frac{3}{x}} = 3$$

$$\frac{1}{\text{highest power denom}}$$

If the degree of the numerator = degree of the denominator the limit is the ratio of the leading coefficients.

$\frac{1}{2}$

b) $\lim\limits_{x \to -\infty} \dfrac{1x^2+1}{2x^2+3x-1}$

c) $\lim\limits_{x\to\infty} \dfrac{x+7}{2x^2+3x-1} \cdot \dfrac{\frac{1}{x^2}}{\frac{1}{x^2}}$

$$\lim_{x\to\infty} \frac{\frac{1}{x} + \frac{7}{x^2}}{2 + \frac{3}{x} - \frac{1}{x^2}} = \frac{0}{2} = 0$$

If the degree of the numerator < degree of the denominator the limit is zero.

d) $\lim\limits_{x\to\infty} \dfrac{x^2+5x}{x-5}$ $\dfrac{\frac{1}{x}}{\frac{1}{x}}$

$$\lim_{x\to\infty} \frac{x+5}{1-\frac{5}{x}} = \infty$$

If the degree of the numerator > degree of the denominator the limit will be either positive or negative infinity.

7. Limits At Infinity Of Functions Containing Radicals

Evaluate

$$\lim_{x \to \infty} \frac{\sqrt{25x^2 + 3x}}{6 - x}$$

Not Rational Function

$$\lim_{x \to \infty} \frac{\sqrt{x^2\left(25 + \frac{3}{x}\right)}}{x\left(\frac{6}{x} - 1\right)}$$

$$\lim_{x\to\infty} \frac{\sqrt{25x^2}}{-x}$$

$$\lim_{x\to\infty} \frac{5\sqrt{x^2}}{-x}$$

$$\lim_{x\to\infty} \frac{5|x|}{-x} = \lim_{x\to\infty} \frac{5\cancel{x}}{-\cancel{x}} = -5$$

Your Turn

Evaluate

$$\lim_{x \to -\infty} \frac{\sqrt{16x^2 - 7}}{x + 5}$$

$$\lim_{x \to -\infty} \frac{\sqrt{x^2\left(16 - \frac{7}{x^2}\right)}}{x\left(1 + \frac{5}{x}\right)}$$

$$\lim_{x \to -\infty} \frac{4\sqrt{x^2}}{x} = \lim_{x \to -\infty} \frac{4|x|}{x}$$

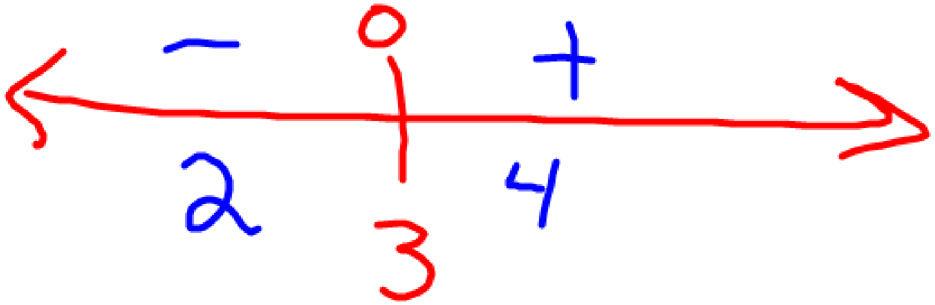
$$\lim_{x \to -\infty} \frac{4(-\cancel{x})}{\cancel{x}} = -4$$

8. Limits Involving Absolute Value

Evaluate the following limits:

a) $\lim_{z\to 3^-} \dfrac{z-3}{|z-3|}$ $= \lim_{z\to 3^-} \dfrac{\overset{1}{\cancel{(z-3)}}}{-1\cancel{(z-3)}} = -1$

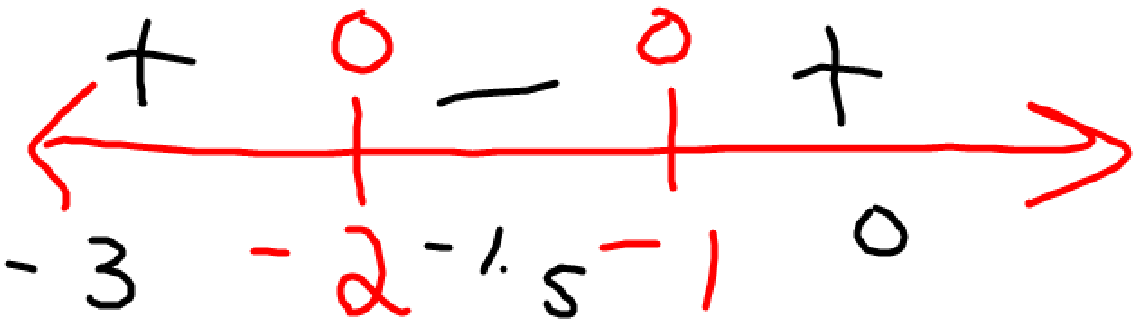
$z-3$



b) $\displaystyle\lim_{x\to -2^-} \frac{\left|x^2+3x+2\right|}{x+2}$

$$= \lim_{x\to -2^-} \frac{\left|(x+2)(x+1)\right|}{(x+2)}$$

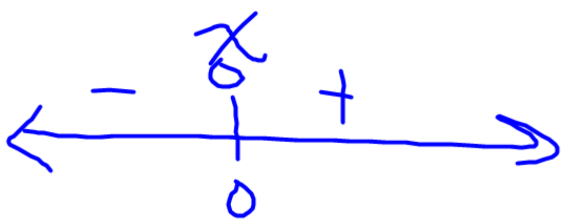
$(x+2)(x+1)$



$$= \lim_{x \to -2^-} \frac{\cancel{(x+2)}(x+1)}{\cancel{(x+2)}} = -2+1 = -1$$

Assignment Page 139
#'s 47-50, 52-56,58-60

$$\lim_{x \to 0^+} \frac{|x|}{x} = \lim_{x \to 0^+} \frac{x}{x} = 1$$



9. Special Trigonometric Limits

We need to commit the following trigonometric limits to memory.

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

$$\lim_{x \to 0} \frac{x}{\sin x} = 1$$

$$\lim_{x \to 0} \frac{1 - \cos x}{x} = 0$$

$$\lim_{x \to 0} \frac{x}{1 - \cos x} = 0$$

Ex.1 Find the limit of the following:

$a)\ \lim_{x \to 0} \frac{\sin x}{2x}$

$b)\ \lim_{x \to 0} \frac{\sin 2x}{x}$

$$\lim_{x \to 0} \left[\left(\frac{1}{2}\right)\left(\frac{\sin x}{x}\right) \right]$$

$$\lim_{x \to 0} \frac{1}{2} \cdot \lim_{x \to 0} \frac{\sin x}{x}$$

$$\frac{1}{2} \cdot (1) = \frac{1}{2}$$

$$= \lim_{x \to 0} 2 \cdot \frac{\sin 2x}{2 \cdot x}$$

$$= \left(\lim_{x \to 0} 2\right) \cdot \left(\lim_{x \to 0} \frac{\sin \cancel{2x}}{\cancel{2x}}\right)$$

$$2 \cdot (1)$$

$$= 2$$

$$c)\ \lim_{x\to 0}\frac{\sin 7x}{\sin 4x}$$

$$d)\ \lim_{x\to \frac{\pi}{6}}\frac{\sin x}{4x}$$

$$\lim_{x\to 0}\frac{(\sin 7x)\frac{7x}{7x}}{(\sin 4x)\frac{4x}{4x}}$$

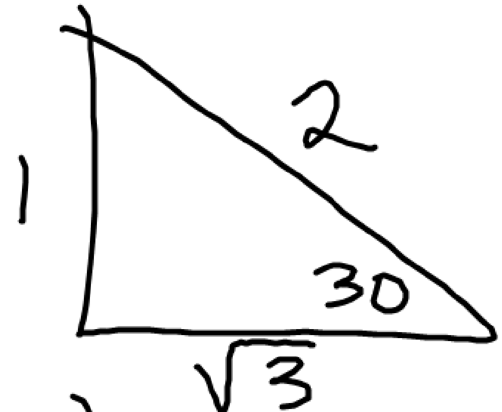
$$= \lim_{x\to 0}\frac{7\cancel{x}\left(\frac{\sin 7x}{7x}\right)}{4\cancel{x}\left(\frac{\sin 4x}{4x}\right)}$$

$$\lim_{x\to 0}\left(\frac{7}{4}\right)\cdot\frac{\left(\lim_{x\to 0}\frac{\sin 7x}{7x}\right)}{\lim_{x\to 0}\frac{\sin 4x}{4x}}$$

$$= \frac{7}{4}\cdot\frac{1}{1}$$

$$= 7/4$$

$$\lim_{x \to \pi/6} \frac{\sin x}{4x}$$



$$\lim_{x \to \pi/6} \frac{\sin \pi/6}{4\left(\frac{\pi}{6}\right)} = \frac{\left(\frac{1}{2}\right)}{\left(\frac{4\pi}{6}\right)}$$

$$= \left(\frac{1}{\cancel{2}}\right)\left(\frac{\cancel{6}^{\,3}}{4\pi}\right)$$

$$= \frac{3}{4\pi}$$

$$e)\ \lim_{x\to 0}\frac{5-5\cos x}{x}$$

$$\lim_{x\to 0}\frac{5(1-\cos x)}{x}$$

$$\lim_{x\to 0} 5\cdot\left(\lim_{x\to 0}\frac{1-\cos x}{x}\right)$$

$$= 0$$

$$\lim_{x\to 0} \frac{\tan 3x}{3\tan 2x}$$

$$= \frac{1}{3}\left(\lim_{x\to 0} \frac{\tan 3x}{\tan 2x}\right)$$

$$= \frac{1}{3}\left(\lim_{x\to 0} \frac{\left(\frac{\sin 3x}{\cos 3x}\right)}{\left(\frac{\sin 2x}{\cos 2x}\right)}\right) = \frac{1}{3}\left(\lim_{x\to 0} \frac{\sin 3x}{\cos 3x} \cdot \frac{\cos 2x}{\sin 2x}\right)$$

$$= \frac{1}{3}\left(\lim_{x\to 0} \frac{\overset{3}{\sin 3x}}{\underset{2}{\sin 2x}} \cdot \frac{\overset{1}{\cos 2x}}{\underset{1}{\cos 3x}}\right)$$

$$= \left(\frac{1}{3}\right)\left(\frac{3}{2}\right)\left(\frac{1}{1}\right) = \frac{1}{2}$$

Page 317 Calc 30 text #'s 3,4,5,6,7,8,12

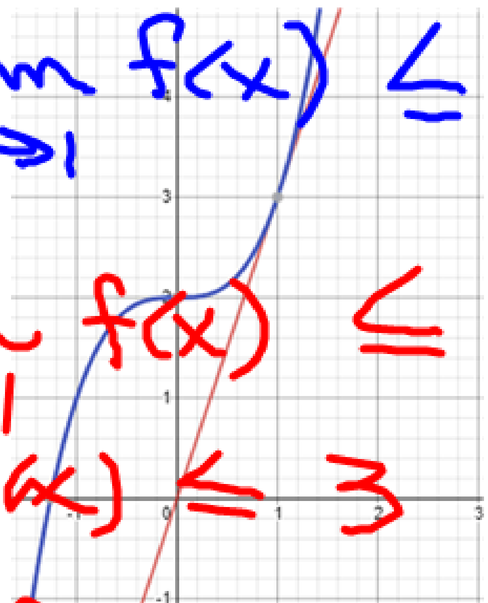
10. The Squeeze/Sandwich Theorem

Intro Video Khan Academy

Use the Squeeze Theorem to find the following limit:

If $3x \le f(x) \le x^3 + 2$ for $0 \le x \le 2$, evaluate: $\lim_{x\to 1} f(x)$

$$\lim_{x\to 1} 3x \le \lim_{x\to 1} f(x) \le \lim_{x\to 1} (x^3+2)$$



$$3(1) \le \lim_{x\to 1} f(x) \le (1)^3+2$$

$$3 \le \lim_{x\to 1} f(x) \le 3$$

$$\therefore \lim_{x\to 1} f(x) = 3$$

Use the Squeeze Theorem to evaluate each of the following:

a) Find $\lim_{x \to 1} f(x)$ given that:

$$4 \le f(x) \le x^2 + 6x - 3 \quad \text{for all } x$$

$$\lim_{x \to 1} 4 \le \lim_{x \to 1} f(x) \le \lim_{x \to 1} (x^2 + 6x - 3)$$

$$4 \le \lim_{x \to 1} f(x) \le (1)^2 + (6)(1) - 3$$

$$4 \le \lim_{x \to 1} f(x) \le 4 \qquad \therefore \lim_{x \to 1} f(x) = 4$$

Bounded Example

The function $f(x) = \cos\left(\frac{1}{x}\right)\sin\left(\frac{x^3-1}{x^4}\right)$ is bounded from below by $g(x) = -\sqrt{1.11 - x^2}$ and from above by $h(x) = \sqrt{1.11 - x^2}$ on the interval [-1, 1]. Is there enough information to determine $\lim\limits_{x \to 0}\left(\cos\left(\frac{1}{x}\right)\sin\left(\frac{x^3-1}{x^4}\right)\right)$?

$$g(x) \leq f(x) \leq h(x)$$

$$-\sqrt{1.11 - x^2} \leq f(x) \leq \sqrt{1.11 - x^2}$$

$$\lim_{x \to 0} -\sqrt{1.11 - x^2} \leq \lim_{x \to 0} f(x) \leq \lim_{x \to 0} \sqrt{1.11 - x^2}$$

$$-\sqrt{1.11} \leq \cancel{\lim_{x \to 0}} f(x) \leq \sqrt{1.11}$$

b) $\lim\limits_{x \to 0} x^4 \sin \frac{3}{x}$

$$-1 \le \sin x \le 1$$

$$x^4(-1) \le x^4 \sin\left(\frac{3}{x}\right) \le 1(x^4)$$

$$-x^4 \le x^4 \sin\left(\frac{3}{x}\right) \le x^4$$

$$\lim_{x \to 0} -x^4 \le \lim_{x \to 0} x^4 \sin\left(\frac{3}{x}\right) \le \lim_{x \to 0} x^4$$

$$0 \le \lim_{x \to 0} x^4 \sin\left(3/x\right) \le 0$$

$$\lim_{x \to 0} x^4 \sin\left(\frac{3}{x}\right) = 0$$

Assignment

Handout #'s 1-7