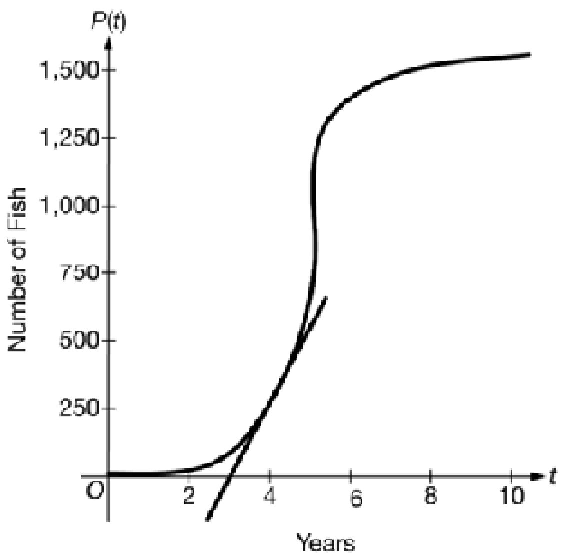
Unit #2 Limits and Continuity

Introducing Calculus, can change occur at a point?

Can change occur at a point part 2

The size of a population of fish in a pond is modeled by the function P, where $P(t)$ gives the number of fish and t gives the number of years after the first year of introduction of the fish to the pond for $0 \leq t \leq 10$. The graph of the function P and the line tangent to P at $t = 4$ are shown above. Which of the following gives the best estimate for the instantaneous rate of change of P at $t = 4$?



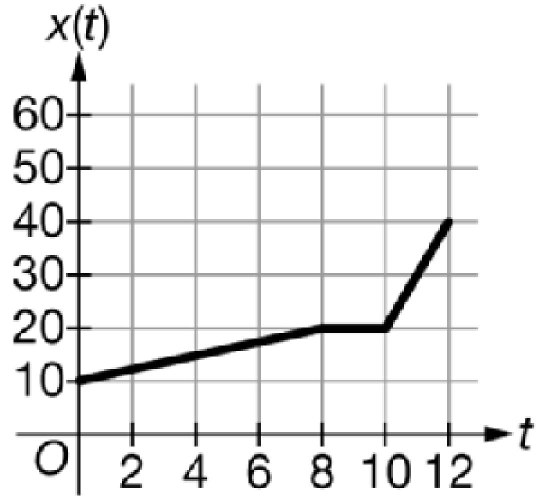
A $P(4)$

B The slope of the line joining $(0, P(0))$ and $(4, P(4))$

C The slope of the line joining $(0, P(0))$ and $(8, P(8))$

D The slope of the line joining $(3.9, P(3.9))$ and $(4.1, P(4.1))$

Your Turn



A particle is moving on the x-axis, and the position of the particle at time t is given by $x(t)$, whose graph is shown above. Which of the following is the best estimate for the speed of the particle at time $t = 6$?

(A) 0

(B) $\frac{5}{4}$

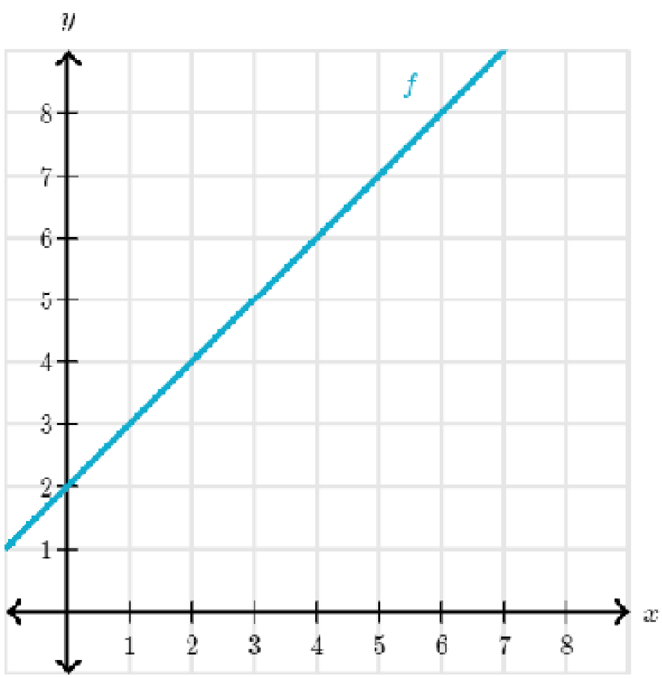
(C) $\frac{35}{12}$

(D) $\frac{10}{3}$

2.1 Limits of a Function

Introduction to Limits

To understand what limits are, let's look at an example. We start with the function $f(x) = x + 2$.



The limit of f at $x = 3$ is the value f approaches as we get closer and closer to $x = 3$. Graphically, this is the y-value we approach when we look at the graph of f and get closer and closer to the point on the graph where $x = 3$.

For example, if we start at the point $(1, 3)$ and move on the graph until we get really close to $x = 3$, then our y-value (i.e. the function's value) gets really close to 5.

Similarly, if we start at $(5, 7)$ and move to the left until we get really close to $x = 3$, the y-value again will be really close to 5.

$$\lim_{x \to 3} f(x) = 5$$

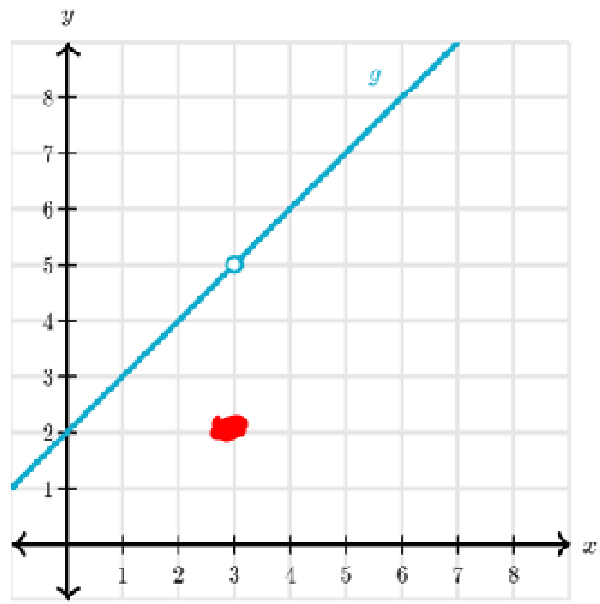
For these reasons we say that **the limit of f at $x = 3$ is 5.**

You might be asking yourselves what's the difference between the limit of f at $x = 3$ and the *value* of f at $x = 3$, i.e. $f(3)$.

So yes, the limit of $f(x) = x + 2$ at $x = 3$ is equal to $f(3)$, but this isn't always the case. To understand this, let's look at function g. This function is the same as f in every way except that it's undefined at $x = 3$.

$\lim_{x \to 3} f(x) = 5$

$f(3) = 2$



Just like f, the limit of g at $x = 3$ is 5. That's because we can still get very very close to $x = 3$ and the function's values will get very very close to 5.

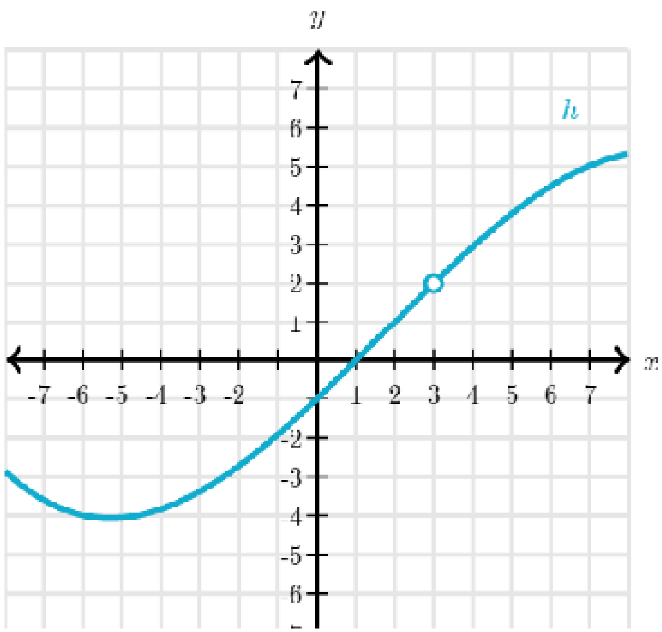
So the limit of g at $x = 3$ is equal to 5, but the value of g at $x = 3$ is undefined! They are not the same!

That's the beauty of limits: they don't depend on the actual value of the function at the limit. They describe how the function behaves when it gets close to the limit.

Reading Limits Off of Graphs

PROBLEM 1

This is the graph of h.

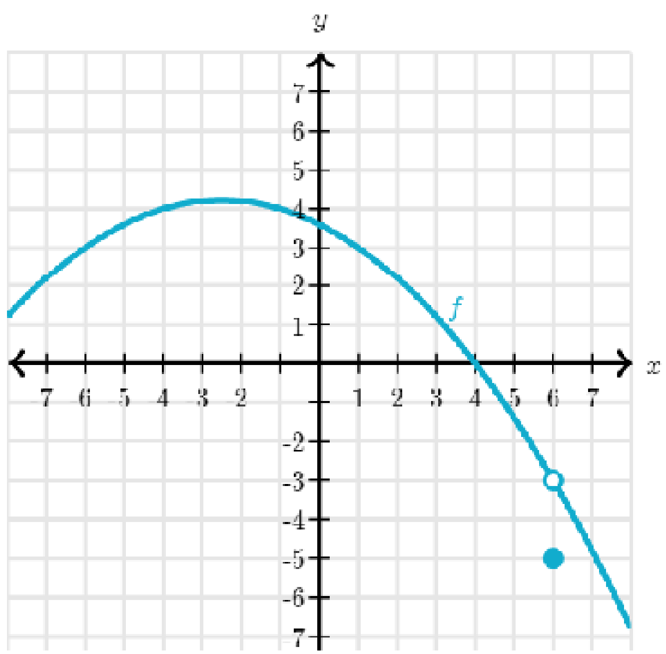


What is a reasonable estimate for the limit of h at $x = 3$?

$= 2$

PROBLEM 2

This is the graph of f.



What is a reasonable estimate for $\lim_{x \to 6} f(x)$?

-3

PROBLEM 3

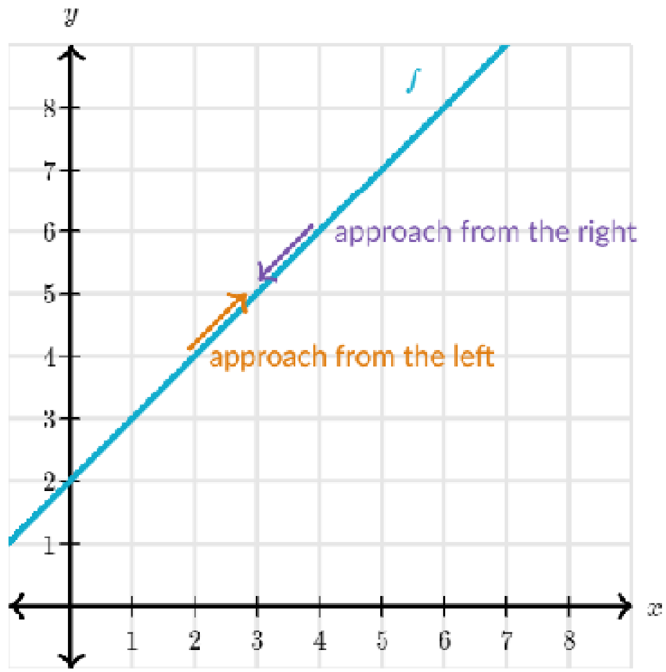
Which expression represents the limit of x^2 as x approaches 5?

Choose **1** answer:

- (A) $\lim 5^2$
- (B) $\lim\limits_{x^2 \to 5}$
- (C) $\lim\limits_{x \to 5} x^2$
- (D) $\lim\limits_{x \to 25} x$

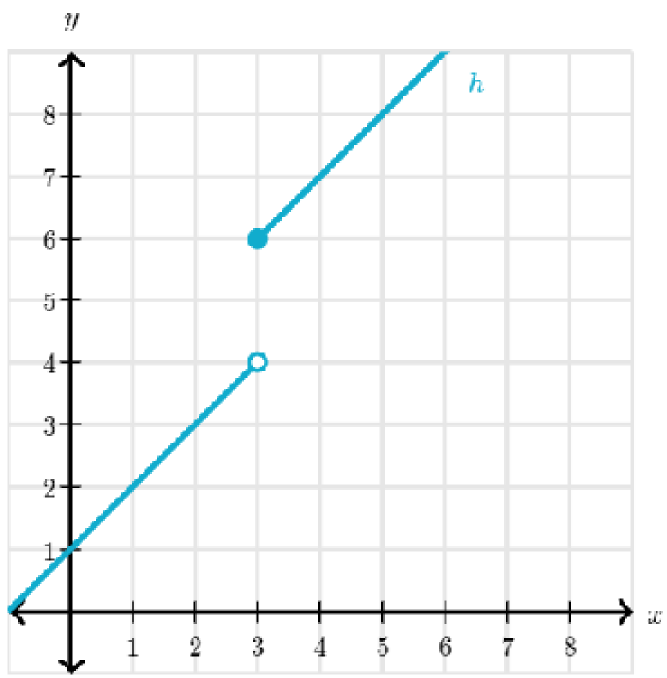
A limit must be the same from both sides.

Coming back to $f(x) = x + 2$ and $\lim_{x \to 3} f(x)$, we can see how 5 is approached whether the x-values increase towards 3 (this is called "approaching from the left") or whether they decrease towards 3 (this is called "approaching from the right").



Now take, for example, function h. The y-value we approach as the x-values approach $x = 3$ depends on whether we do this from the left or from the right.

$$\lim_{x \to 3^-} h(x) = 4$$



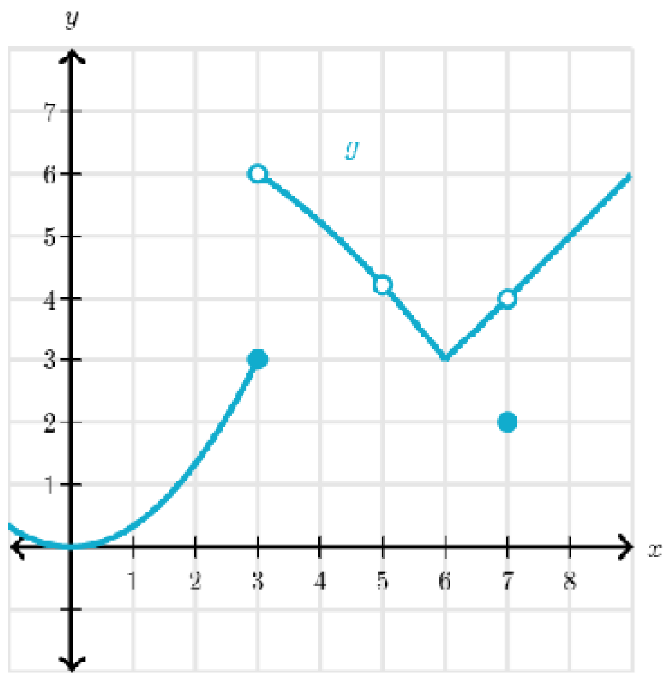
$$\lim_{x \to 3^+} h(x) = 6$$

$$\lim_{x \to 3} h(x) = DNE$$

When a limit doesn't approach the same value from both sides, we say that the limit doesn't exist.

PROBLEM 5

This is the graph of function g.



Which of the limits exists?

Choose all answers that apply:

(A) ~~$\lim_{x \to 3} g(x)$~~

(B) $\lim_{x \to 5} g(x)$

(C) $\lim_{x \to 6} g(x)$

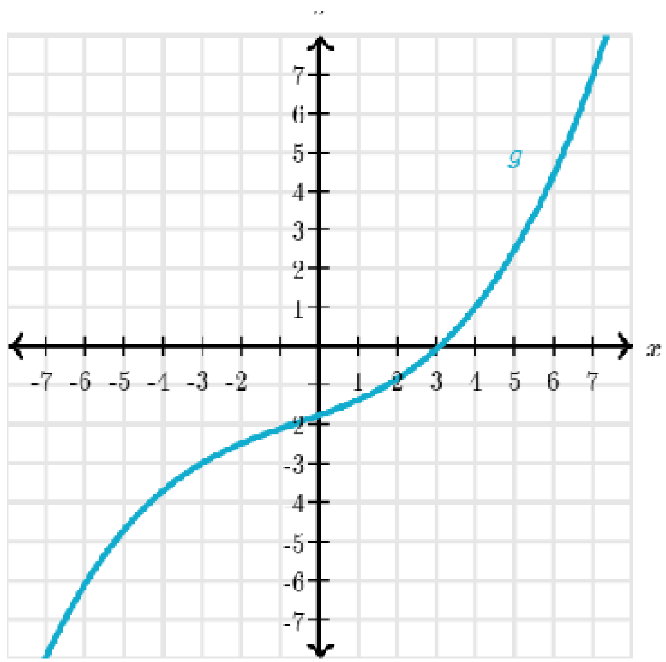
(D) $\lim_{x \to 7} g(x)$

Your Turn

PROBLEM 1

What is a reasonable estimate for $\lim_{x \to 1} g(x)$?

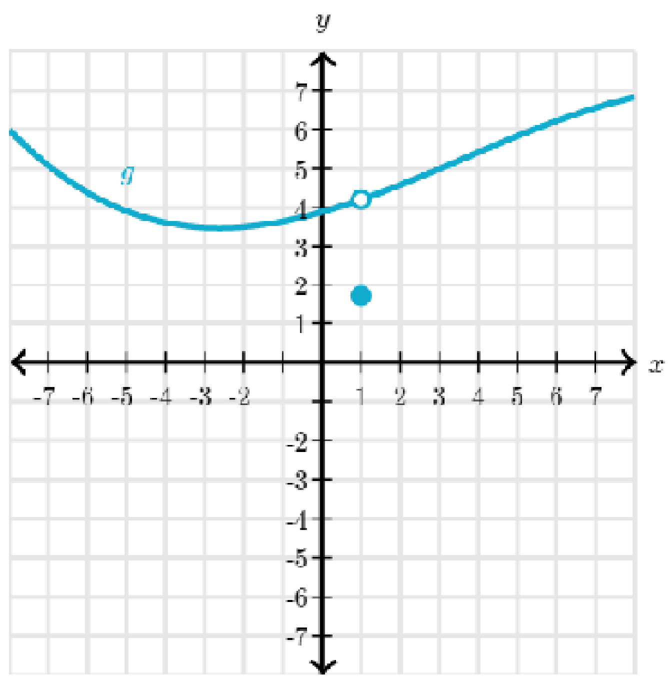
$= -5/4$



PROBLEM 2

What is a reasonable estimate for $\lim_{x \to 1} g(x)$?

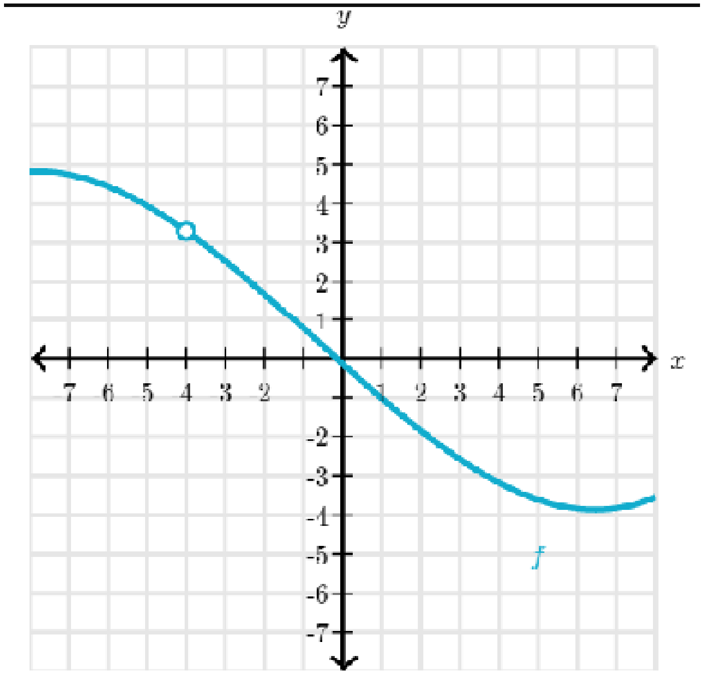
4



PROBLEM 3

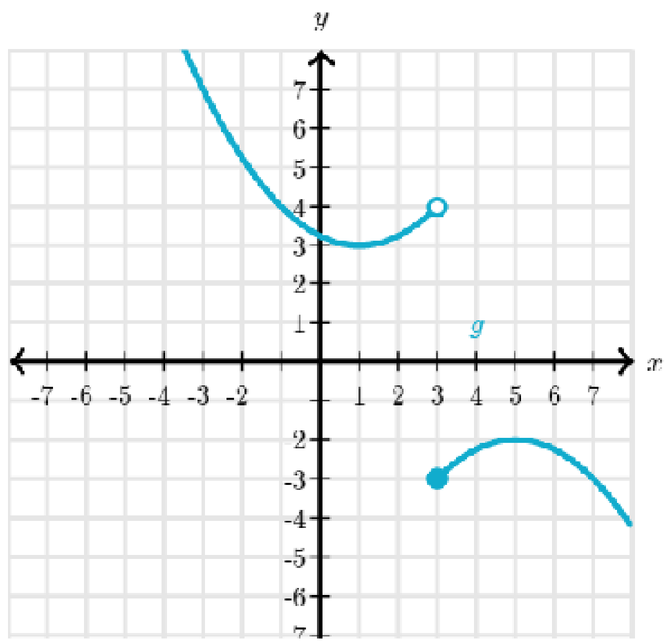
What is a reasonable estimate for $\lim_{x \to -4} f(x)$?

3



PROBLEM 4

What is a reasonable estimate for $\lim_{x \to 3} g(x)$?

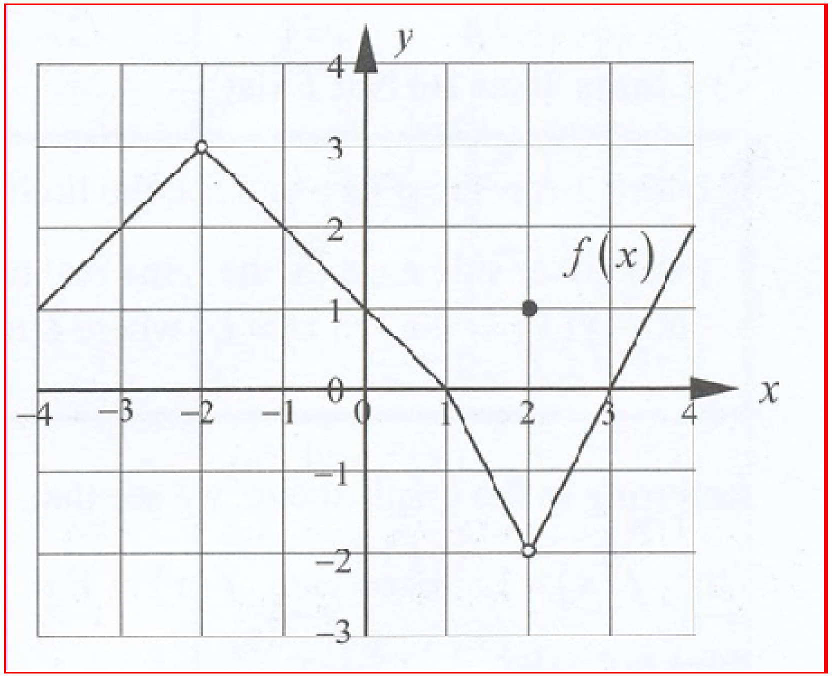


DNE

By examining the graph determine the following limits:

(a) $\lim_{x\to 0} f(x)$ 1 (b) $\lim_{x\to 3} f(x)$ 0 (c) $\lim_{x\to -1} f(x)$ 2

(d) $\lim_{x\to -2} f(x)$ 3 (e) $\lim_{x\to 2} f(x)$ -2 (f) $\lim_{x\to -3} f(x)$ 2



One Sides Limits

Sometimes the values of a function *f* tend to different limits as x approaches a number from opposite sides

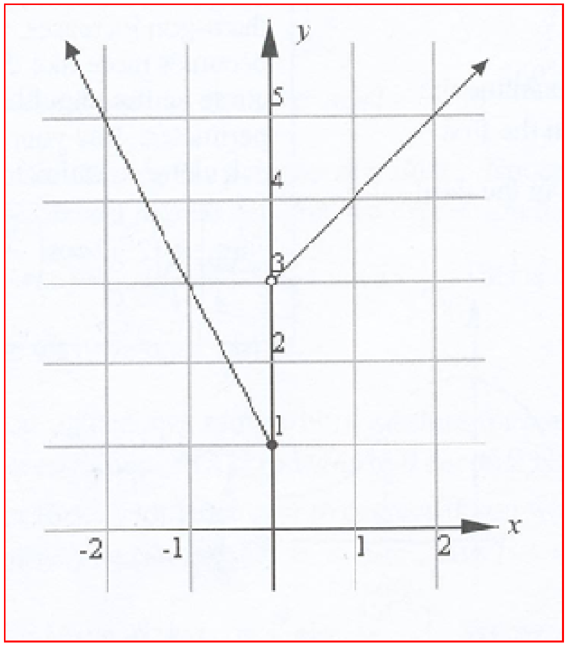
$$\lim_{x \to c^+} f(x) = \text{right hand limit}$$

$$\lim_{x \to c^-} f(x) = \text{left hand limit}$$

Definition: A function f(x) has a limit as x approaches c if and only if the right hand limit and the left hand limits at c exist and are equal.

In order for $\lim_{x \to b} f(x)$ to exist, the limit as you approach b from either side must be the same real number. That is $\lim_{x \to b^+} f(x) = \lim_{x \to b^-} f(x) = L$, where L is a real number.

Example 1: Evaluate the following limits.



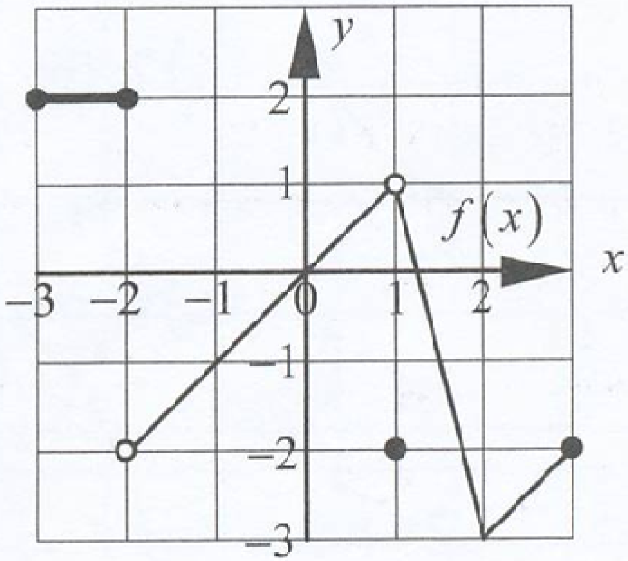
$$\lim_{x \to 0^+} f(x) = 3$$

$$\lim_{x \to 0} f(x) = 1$$

$$\lim_{x \to 0} f(x) = \text{DNE}$$

Example 2: Evaluate the following limits.

By referring to the graph of $f(x)$ below, determine each of the following.



(a) $\lim_{x \to 2} f(x)$ -3

(b) $\lim_{x \to 0} f(x)$ 0

(c) $\lim_{x \to -2.5} f(x)$ 2

(d) $\lim_{x \to 1} f(x)$ 1

(e) $f(1)$ -2

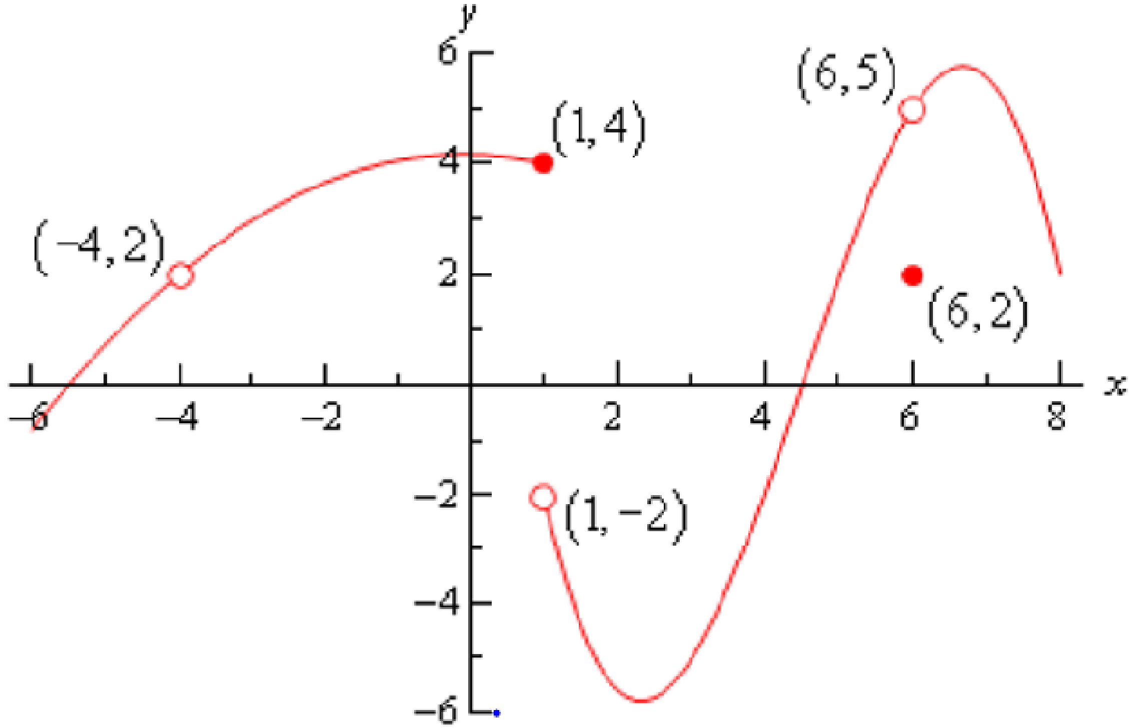
(f) $f(-2)$ 2

(g) $\lim_{x \to -2^+} f(x)$ -2

(h) $\lim_{x \to -2^-} f(x)$ 2

(i) $\lim_{x \to -2} f(x)$ DNE

Example 4 Given the following graph,



compute each of the following.

(a) $f(-4)$ DNE

(b) $\lim\limits_{x \to -4^-} f(x)$ 2

(c) $\lim\limits_{x \to -4^+} f(x)$ 2

(d) $\lim\limits_{x \to -4} f(x)$ 2

(e) $f(1)$ 4

(f) $\lim\limits_{x \to 1^-} f(x)$ 4

(g) $\lim\limits_{x \to 1^+} f(x)$ −2

(h) $\lim\limits_{x \to 1} f(x)$ DNE

(i) $f(6)$ 2

(j) $\lim\limits_{x \to 6^-} f(x)$ 5

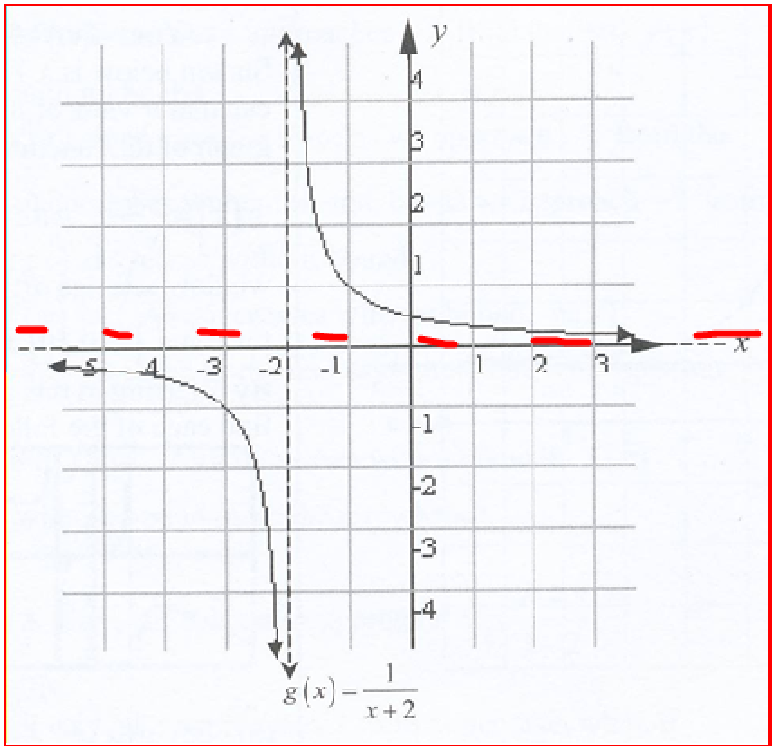
(k) $\lim\limits_{x \to 6^+} f(x)$ 5

(l) $\lim\limits_{x \to 6} f(x)$ 5

Assignment
Page 123 (Calc 30 Text)
#'s 8, 9
Page 62 (AP Calc Text)
#'s 1-6

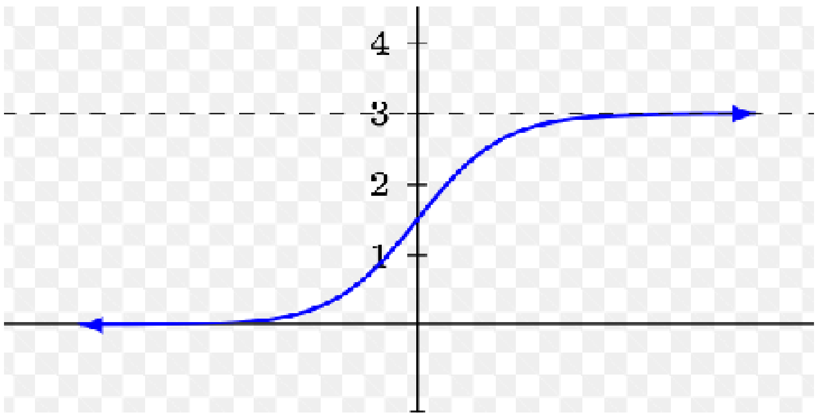
Reading Off Of Graphs One Side Limits Involving Vertical and Horizontal Asymptotes

Limits at Infinity



$$\lim_{x\to\infty} \frac{1}{x+2} = 0$$

$$\lim_{x\to-\infty} \frac{1}{x+2} = 0$$



$y = f(x)$

$\lim_{x\to\infty} f(x) = 3$

$\lim_{x\to-\infty} f(x) = 0$

Definition: The line y=b is a **horizontal asymptote** of a function if:

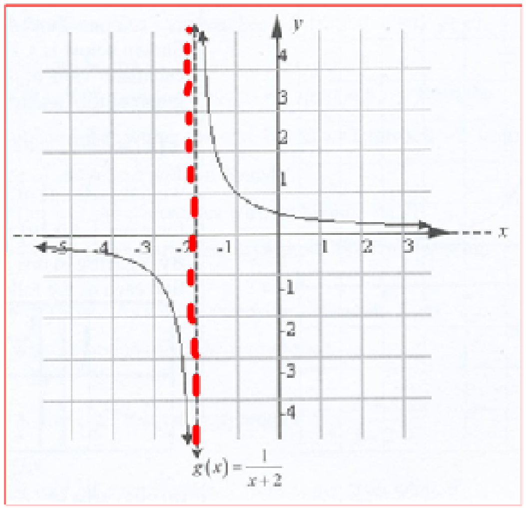
$$\lim_{x\to\infty} f(x) = b \text{ or } \lim_{x\to-\infty} f(x) = b$$

equation HA

$y = b$

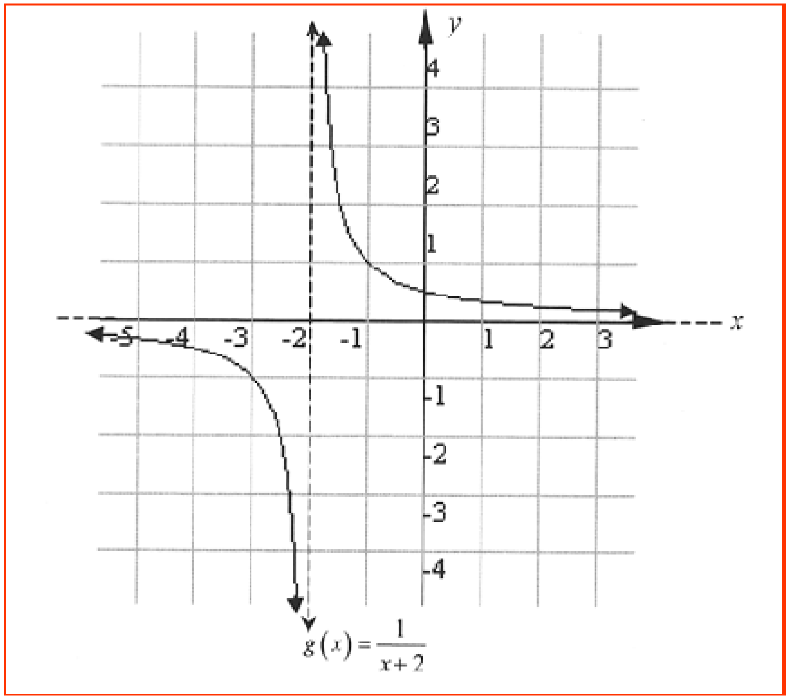
Now lets go back to our function:

$$\boxed{g(x) = \frac{1}{x+2}}$$



$$\boxed{\lim_{x \to -2^+} \left(\frac{1}{x+2} \right)} \quad \infty$$

$$\boxed{\lim_{x \to -2^-} \left(\frac{1}{x+2} \right)} \quad -\infty$$

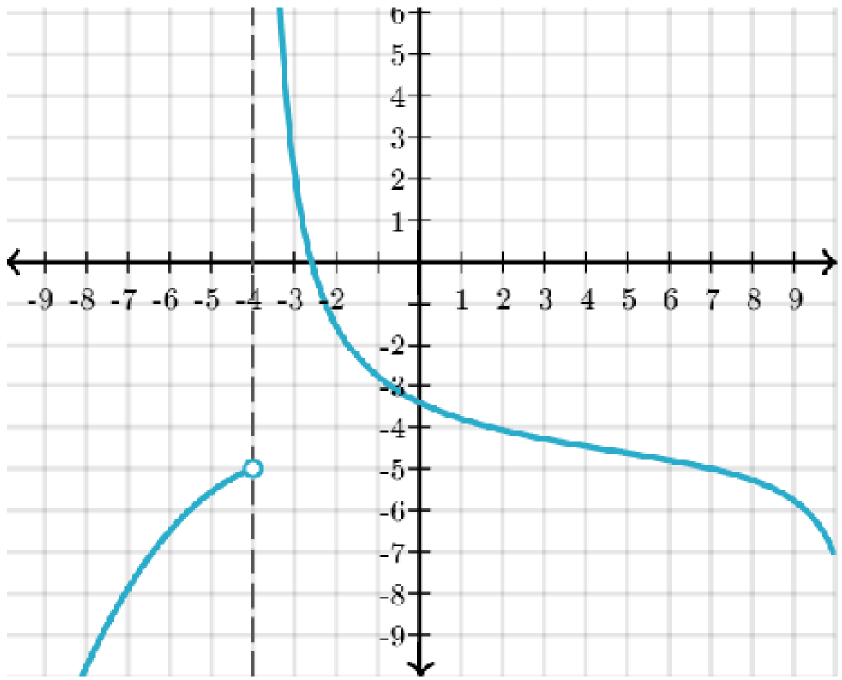


$\lim\limits_{x \to -2^-} f(x)$ $-\infty$

$\lim\limits_{x \to -2^+} f(x)$ ∞

$\lim\limits_{x \to -2} f(x)$ DNE

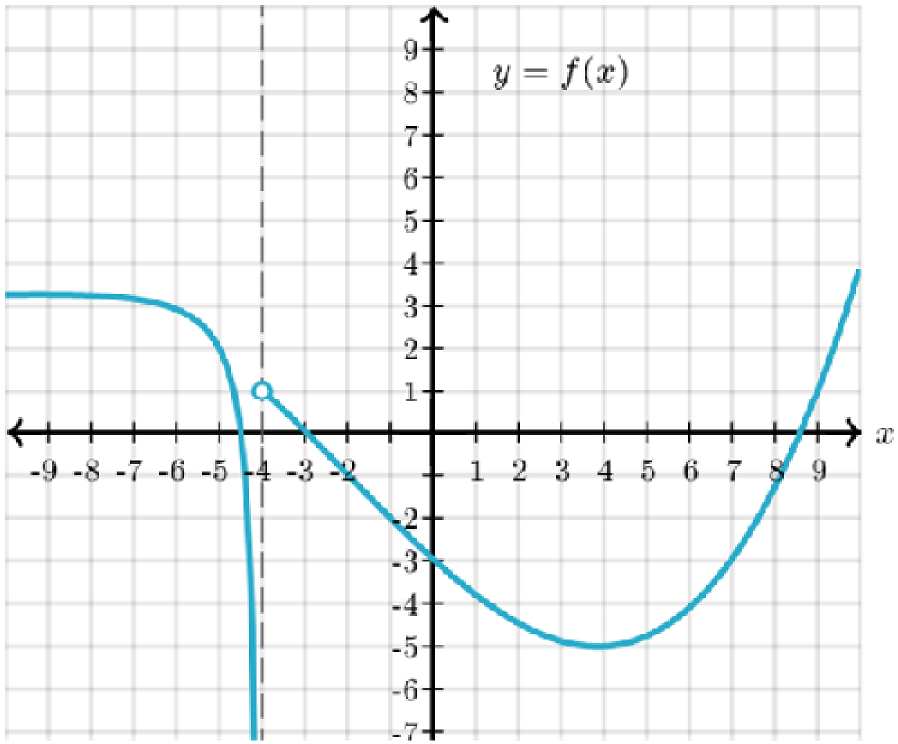
Function h is graphed.



What appears to be the value of $\lim_{x \to -4^+} h(x)$?

∞

$$\lim_{x \to \infty} f(x) = 2$$



What appears to be the value of $\lim\limits_{x \to -4^+} f(x)$?

1

$$\lim_{x \to -4^-} f(x) = -\infty$$

For the function $h(x)$ whose graph is shown, find each of the following.

(a) $\lim_{x \to 0} h(x)$ 0

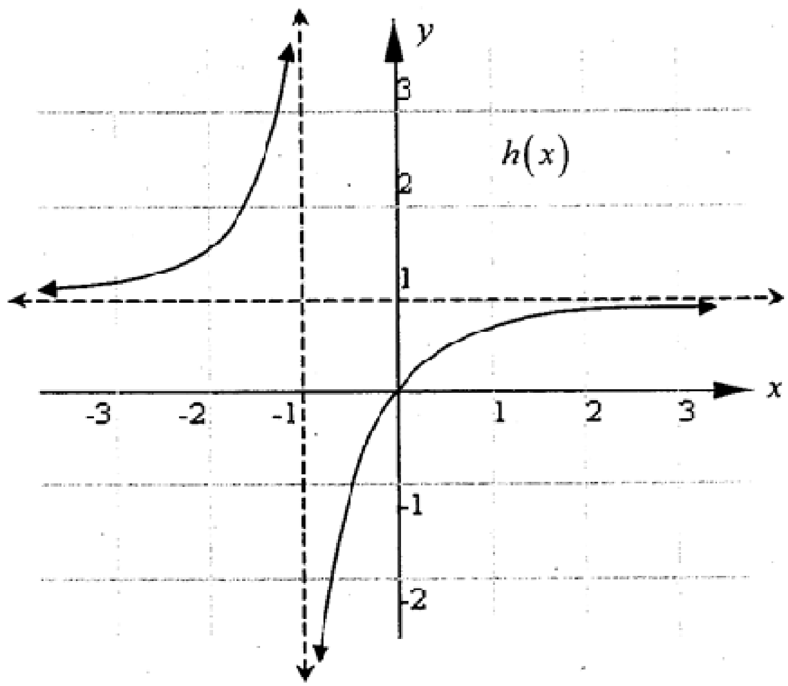
(b) $\lim_{x \to -1^-} h(x)$ ∞

(c) $\lim_{x \to -1^+} h(x)$ $-\infty$

(d) $\lim_{x \to -1} h(x)$ DNE

(e) $\lim_{x \to \infty} h(x)$ 1

(f) $\lim_{x \to -\infty} h(x)$ 1



For the function $j(x)$ whose graph is shown, find each of the following.

(a) $\lim_{x\to -1^+} j(x)$ ∞

(b) $\lim_{x\to -1^-} j(x)$ ∞

(c) $\lim_{x\to -1} j(x)$ ∞

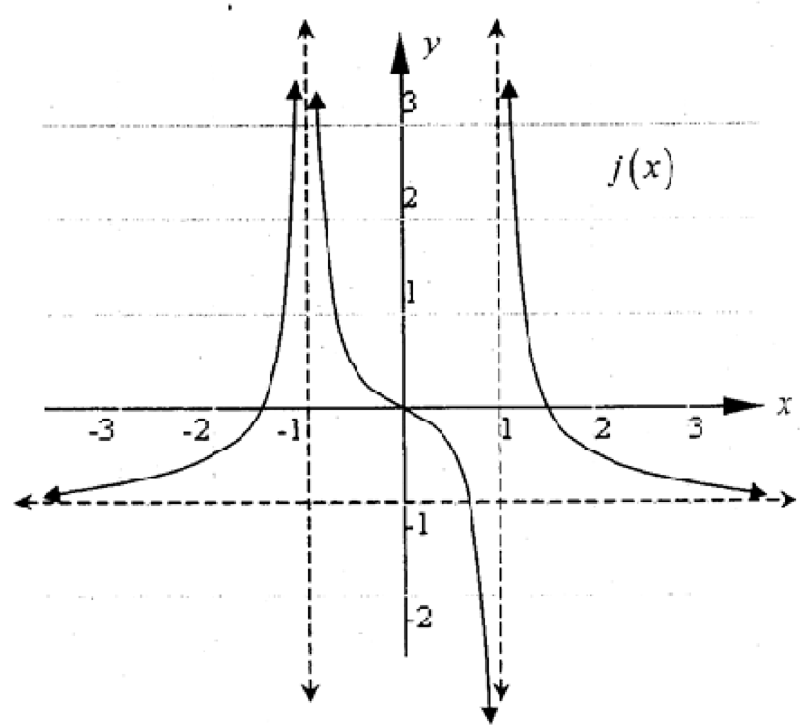
(d) $\lim_{x\to 1^+} j(x)$ ∞

(e) $\lim_{x\to 1^-} j(x)$ $-\infty$

(f) $\lim_{x\to 1} j(x)$ DNE

(g) $\lim_{x\to \infty} j(x)$ -1

(h) $\lim_{x\to -\infty} j(x)$ -1



Definition: The line x = a is a **vertical asymptote** of a function if:

$$\lim_{x \to a^+} f(x) = \pm\infty$$

or

$$\lim_{x \to a^-} f(x) = \pm\infty$$

Sketching Graphs With Certain Conditions Involving Infinity

Sketch the graph of the function $y = f(x)$ that satisfies all given conditions.
Indicate all vertical and horizontal asymptotes as dotted lines.

$$\lim_{x\to 1} f(x) = 3$$

$$\lim_{x\to 2^-} f(x) = \infty$$

$$\lim_{x\to -\infty} f(x) = 0$$

$$\lim_{x\to 2^+} f(x) = -\infty$$

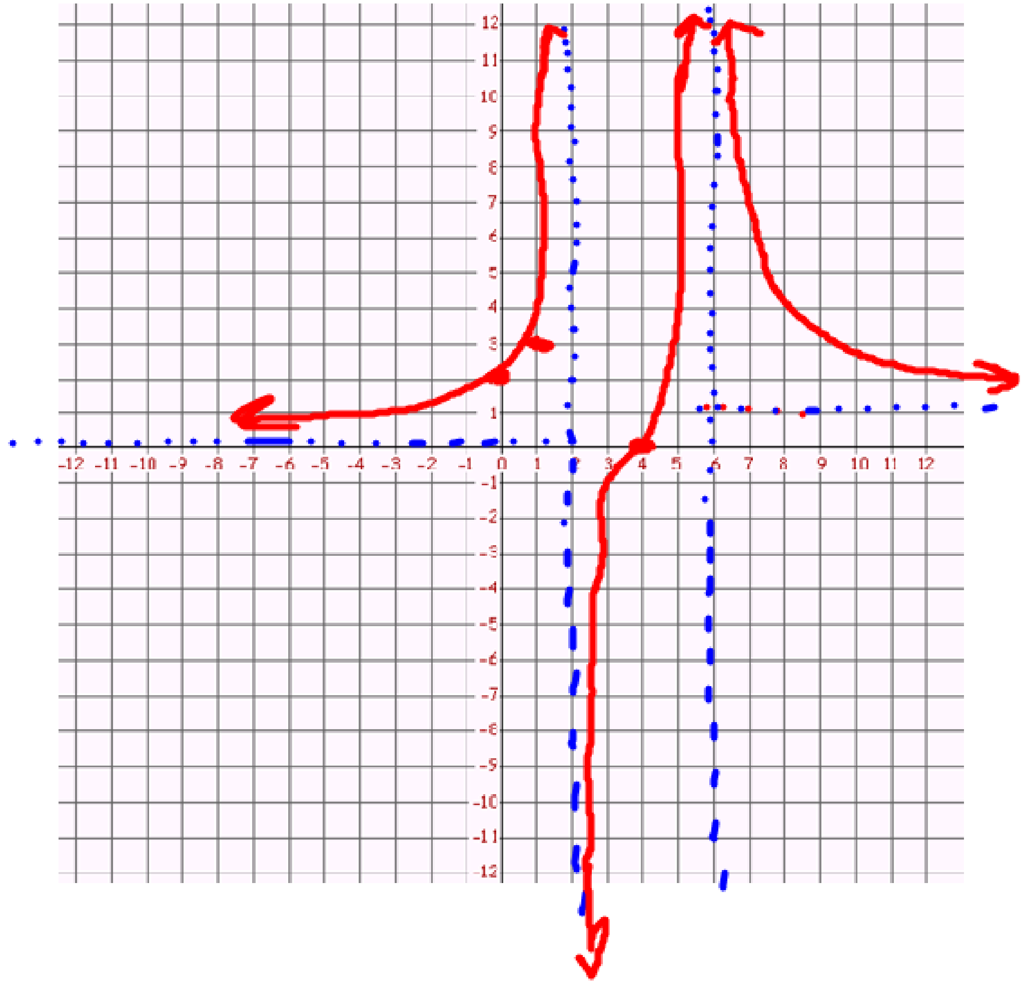
$$f(4) = 0$$

$$\lim_{x\to 6^-} f(x) = \infty$$

$$\lim_{x\to 6^+} f(x) = \infty$$

$$\lim_{x\to \infty} f(x) = 1$$

$$f(0) = 2$$



Assignment
Handout
Page 72 (AP Calculus Text)
#'s 49-50
Matching Game

Your turn to try!

Example 2 By examining the graph of $f(x)$ below, determine each of the following.

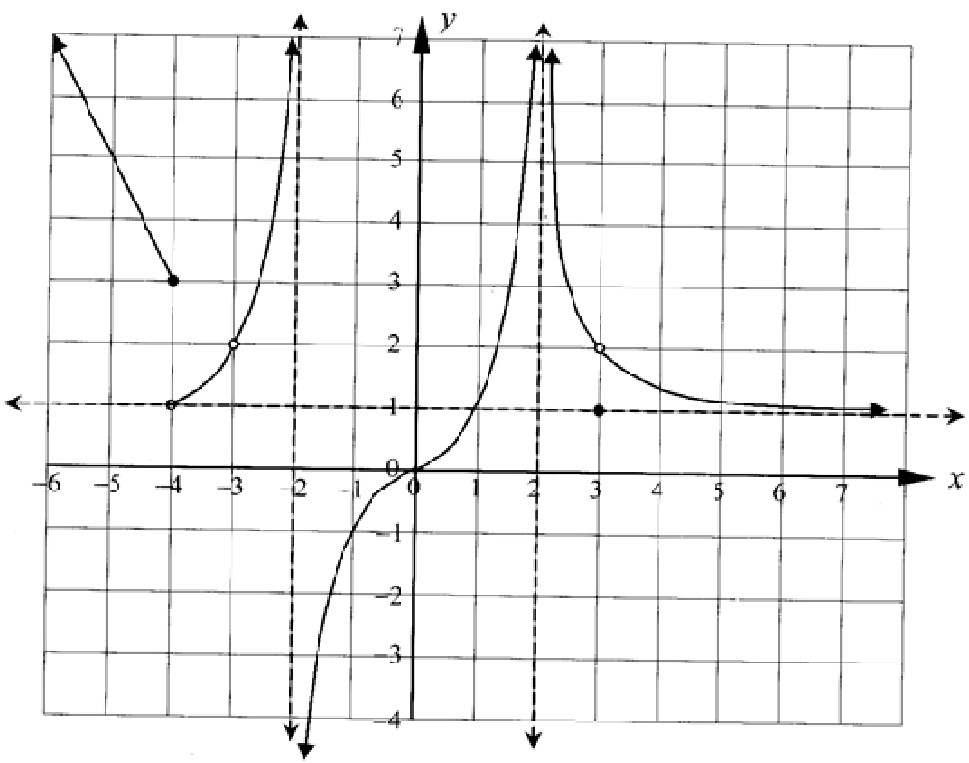
(a) $f(-6)$ 7 (b) $f(0)$ 0 (c) $f(3)$ 1 (d) $f(-4)$ 3 (e) $f(-3)$ DNE

(f) $\lim_{x\to 1} f(x)$ 1 (g) $\lim_{x\to -5} f(x)$ 3 (h) $\lim_{x\to 0} f(x)$ 0 (i) $\lim f(x)$ 2

(j) $\lim_{x\to -4^+} f(x)$ 1 (k) $\lim_{x\to -4^-} f(x)$ 3 (l) $\lim_{x\to -4} f(x)$ DNE (m) $\lim_{x\to 2^+} f(x)$ ∞

(n) $\lim_{x\to 2^-} f(x)$ ∞ (o) $\lim_{x\to 2} f(x)$ ∞ (p) $\lim_{x\to -2^+} f(x)$ $-\infty$ (q) $\lim_{x\to -2^-} f(x)$ ∞

(r) $\lim_{x\to -2} f(x)$ DNE (s) $\lim_{x\to\infty} f(x)$ 1 (t) $\lim_{x\to -\infty} f(x)$ ∞



Evaluating Limits Off of Tables

The function g is defined over the real numbers. This table gives select values of g.

x	$g(x)$
4	3.37
4.9	3.5
4.99	3.66
4.999	3.68
5	6.37
5.001	3.68
5.01	3.7
5.1	3.84
6	3.97

What is a reasonable estimate for $\lim_{x \to 5} g(x)$?

3.68

PROBLEM 1

Three students were given a function f and asked to estimate $\lim_{x \to 2} f(x)$. Each student created a table (shown below).

Each table is accurate, but which one is the best for approximating the limit?

(A)

A	x	-1	0	1	2	3	4
	$f(x)$	0.2	1.3	2.9	3.25	2.9	1.3

3.25

(B)

B	x	1.9	1.99	1.999	2.001	2.01	2.1
	$f(x)$	3.32	3.264	3.251	3.249	3.242	3.31

(C)

C	x	2.001	2.01	2.1	2.25	2.5	3
	$f(x)$	3.249	3.242	3.31	3.2	3.05	2.9

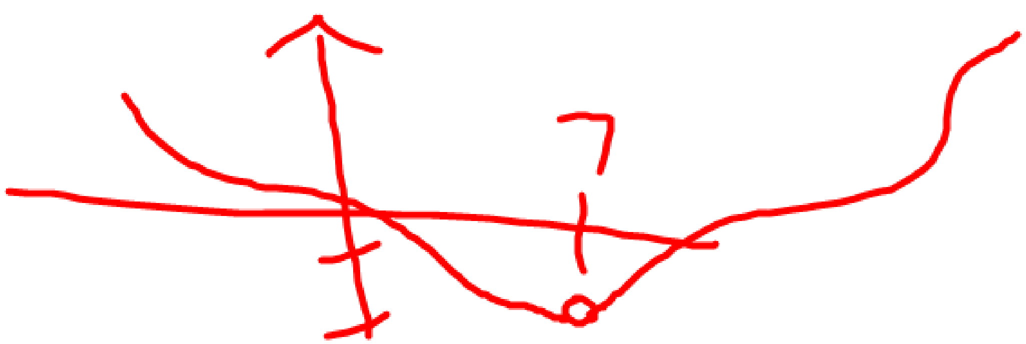
Your Turn

A student created a table to help them reason about $\lim\limits_{x \to 7} g(x)$.

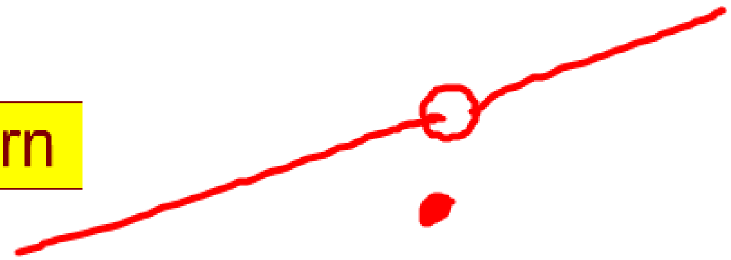
x	6	6.99	6.9999	7	7.0001	7.01	8
$g(x)$	-3.41	-1.94	-1.9252	undefined	-1.9248	-1.91	0.46

Based on the table, what can you reasonably conclude about the limit?

-1.92



Your Turn



PROBLEM 4

The table gives a few values of function f. The function is increasing everywhere except at $x = 5$, and $\lim_{x \to 5} f(x)$ exists.

x	2	3	4	5	6	7	8
$f(x)$	3.7	4.3	4.9	4.8	5.6	6.2	6.9

Which is a reasonable estimate for $\lim_{x \to 5} f(x)$?

Choose 1 answer:

(A) 4.6

(B) 4.8

(C) 5.3

(D) 5.6

Evaluating One Side Limits Off Tables

Ex. 1

The function f is defined over the real numbers. This table gives select values of f.

x	0.9	0.99	0.999	1	1.001	1.01	1.1
$f(x)$	2.5	2.1	2.02	5	-0.99	-0.92	-0.81

What is a reasonable estimate for $\lim_{x \to 1^-} f(x)$**?** 2

Ex. 2

The function f is defined over the real numbers. This table gives select values of f.

x	-2.05	-2.01	-2.002	-2	-1.998	-1.99	-1.95
$f(x)$	-20	-100	-500	-4	-4.016	-4.08	-4.3998

What is a reasonable estimate for $\lim_{x \to -2^-} f(x)$?

$\lim_{x \to -2^+} f(x) = -4$

unbounded

$-\infty$

Assignment
Handout
2.1 Reading Limits Off
Graphs and Tables

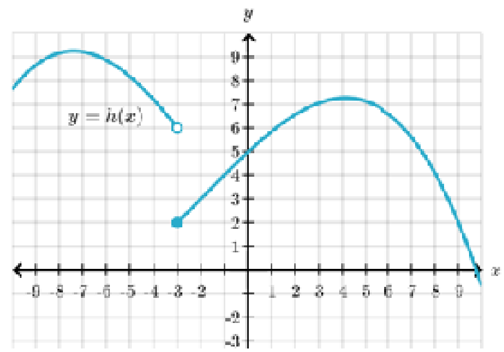
2.1 Reading Limits Off Graphs and Tables

The function f is defined over the real numbers. This table gives a few values of f.

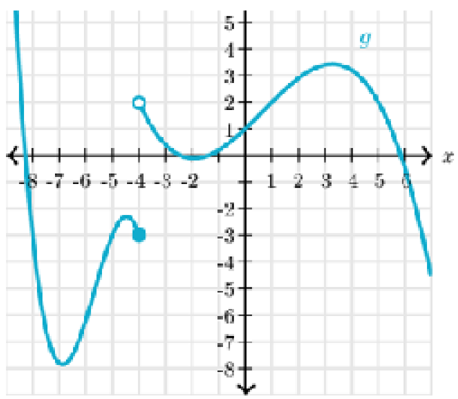
x	$f(x)$
-7.1	4.37
-7.01	4.18
-7.001	4.01
-6.999	-4.01
-6.99	-4.18
-6.9	-4.37

What is a reasonable estimate for $\lim_{x \to -7} f(x)$?

Function h is graphed.



What appears to be the value of $\lim_{x \to -3} h(x)$?



What is a reasonable estimate for $\lim_{x \to -4} g(x)$?

The function g is defined over the real numbers. This table gives several values of g.

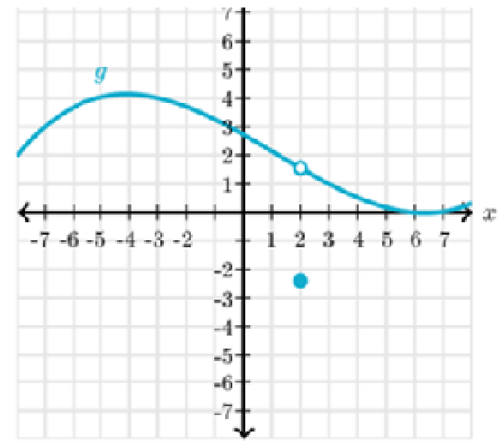
x	3.75	3.9	3.99	3.999	4.001	4.01	4.1	4.25
$g(x)$	-6.7	-6.85	-6.985	-6.999	-7	-6.997	-6.97	-6.73

What is a reasonable answer to the $\lim_{x \to 4} g(x)$?

The function h is defined over the real numbers except at $x = -7$. This table gives select values of h.

x	-7.016	-7.004	-7.001	-7	-6.999	-6.996	-6.984
$h(x)$	4.888	4.972	4.993	undefined	-830	-664	-497

What is a reasonable estimate for $\lim\limits_{x \to -7^+} h(x)$?



What is a reasonable estimate for $\lim\limits_{x \to 2} g(x)$?

Properties of Limits

LIMIT LAWS

Let $f(x)$ and $g(x)$ be defined for all $x \neq a$ over some open interval containing a. Assume that L and M are real numbers such that $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$. Let c be a constant. Then, each of the following statements holds:

Sum law for limits: $\lim_{x \to a} (f(x) + g(x)) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x) = L + M$

Difference law for limits: $\lim_{x \to a} (f(x) - g(x)) = \lim_{x \to a} f(x) - \lim_{x \to a} g(x) = L - M$

Constant multiple law for limits: $\lim_{x \to a} cf(x) = c \cdot \lim_{x \to a} f(x) = cL$

Product law for limits: $\lim_{x \to a} (f(x) \cdot g(x)) = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x) = L \cdot M$

Quotient law for limits: $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \frac{L}{M}$ for $M \neq 0$

Power law for limits: $\lim_{x \to a} (f(x))^n = \left(\lim_{x \to a} f(x)\right)^n = L^n$ for every positive integer n.

Root law for limits: $\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)} = \sqrt[n]{L}$ for all L if n is odd and for $L \geq 0$ if n is even.

Limits of Combined Functions

Ex. 1 Given that $\lim_{x\to a} f(x) = -3$ and $\lim_{x\to a} g(x) = 0$ and $\lim_{x\to a} h(x) = 8$ find the limits if they exist

a) $\lim_{x\to a}\left[h(x) + f(x)\right]$

$\lim_{x\to a} h(x) + \lim_{x\to a} f(x)$

$8 + -3 = 5$

b) $\lim_{x\to a}\left[f(x)\right]^2$

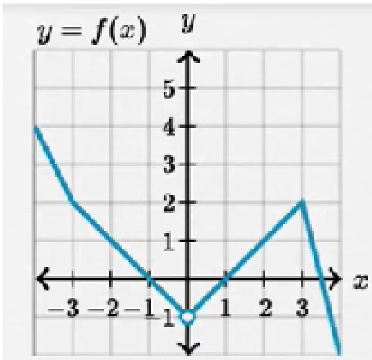
$\left[\lim_{x\to a} f(x)\right]^2$

$(-3)^2 = 9$

c) $\lim_{x\to a} \dfrac{2f(x)}{h(x) - f(x)}$

$$\frac{\lim_{x\to a} 2f(x)}{\lim_{x\to a}\left(h(x) - f(x)\right)} = \frac{2\left(\lim_{x\to a} f(x)\right)}{\lim_{x\to a} h(x) - \lim_{x\to a} f(x)}$$

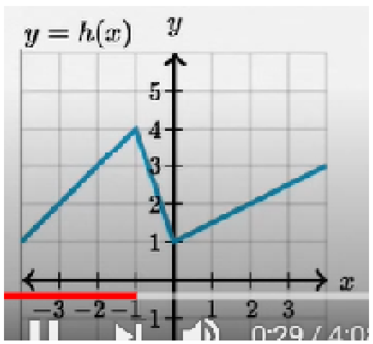
$$= \frac{2(-3)}{8-(-3)} = \frac{-6}{11}$$

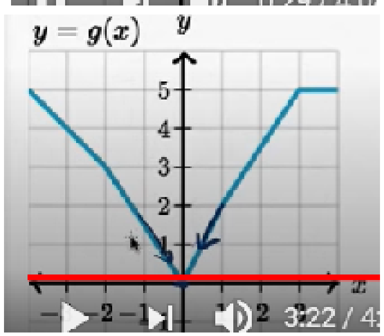


$$\left(\lim_{x\to 0} f(x)\right)\left(\lim_{x\to 0} h(x)\right)$$

$$(-1)(1) = -1$$

$$\lim_{x\to 0}[f(x)h(x)]$$

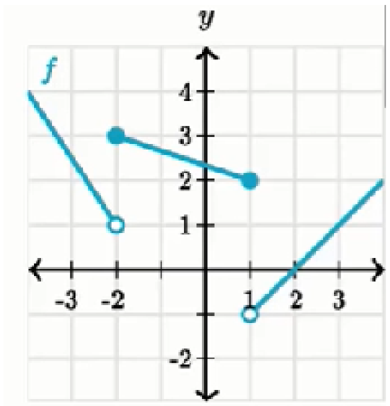




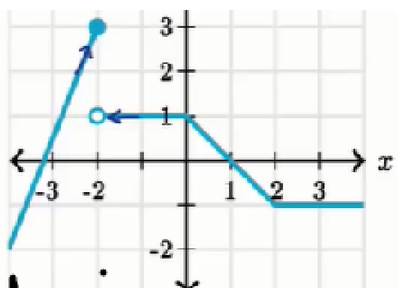
$$\lim_{x\to 0}\frac{h(x)}{g(x)}$$

$$\frac{\lim_{x\to 0} h(x)}{\lim_{x\to 0} g(x)} = \frac{1}{0} \quad \underline{\text{UND}}$$

$$\lim_{x\to -2}(f(x)+g(x))$$



g



$$\lim_{x\to 1}(f(x)+g(x))$$ DNE

$$\lim_{x\to 1^-} f(x)+\lim_{x\to 1^-} g(x)$$

$$2+0=2$$

$$\lim_{x\to 1^+} f(x)+\lim_{x\to 1^+} g(x)$$

$$-1+0=-1$$

$$\lim_{x\to 1}(f(x)g(x)) = 0$$

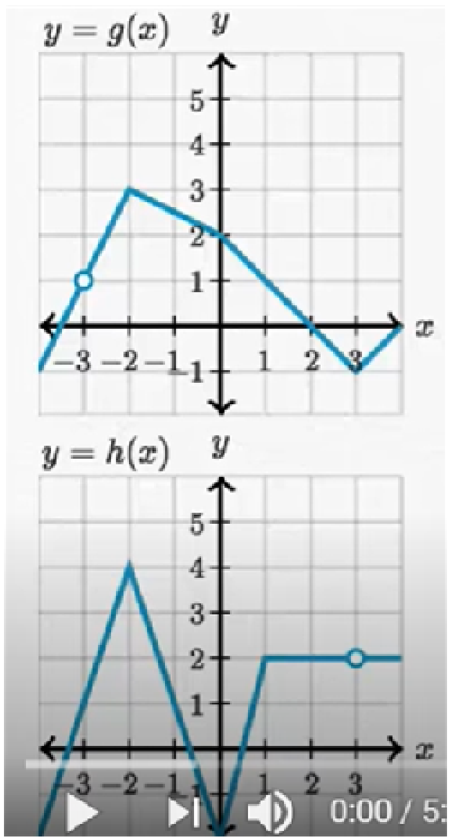
$$\lim_{x\to 1^-} f(x)\cdot\lim_{x\to 1^-} g(x)$$

$$2\cdot 0=0$$

$$\lim_{x\to 1^+} f(x)\cdot\lim_{x\to 1^+} g(x)$$

$$(-1)(0)=0$$

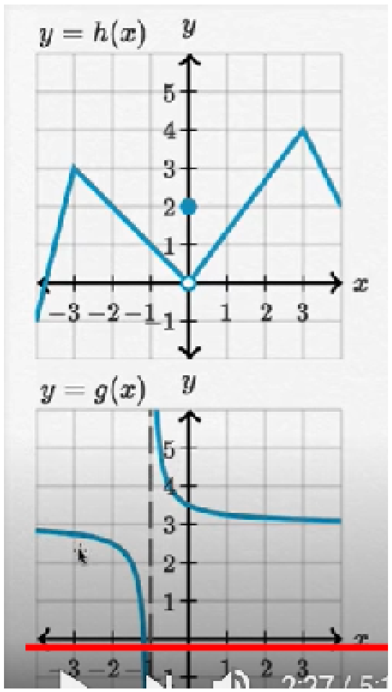
Limits of Composite Functions



$$\boxed{\lim_{x\to 3} g(h(x))}$$

$$g\left(\lim_{x\to 3} h(x)\right)$$

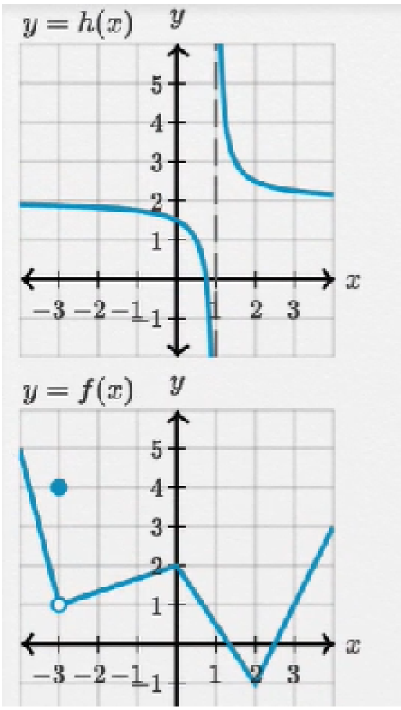
$$g(2) = 0$$



$$\boxed{\lim_{x \to -1} h(g(x))}$$ DNE

$$h\left(\lim_{x \to -1} g(x)\right)$$

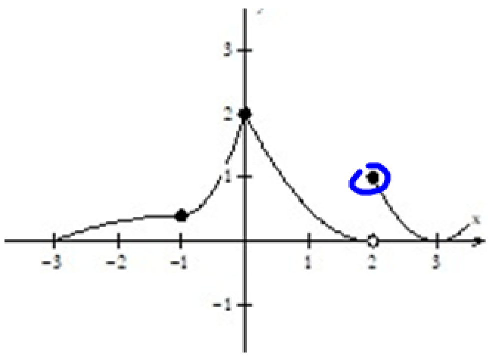
DNE



$$\boxed{\lim_{x \to -3} h(f(x))}$$

$$h\left(\lim_{x \to -3} f(x)\right)$$

$h(1)$ DNE



If you look at

$$\lim_{x \to 0} f\left(f\left(x\right)\right)$$

$$f\left(\lim_{x \to 0} f(x)\right)$$

$$f(2)$$

one side limit

$$= 0$$

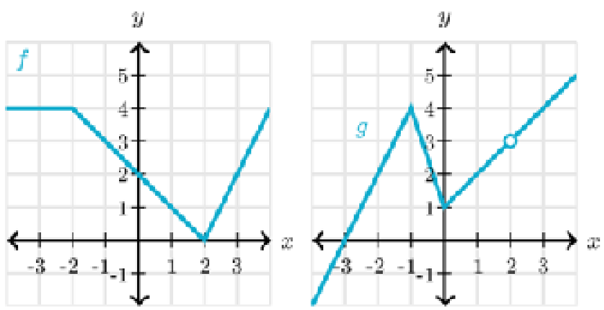
You can see that as x approaches 0, the graph of f(x) approaches 2. However, the graph of f is not continuous at 2, so we have to use one-sided limits. From the graph, as x approaches 0, f(x) approaches 2 from below on both sides, therefore we can think about this as the limit as x approaches 2 from the left of f(x). Which gives us 0. Therefore

$$\lim_{x \to 0} f\left(f\left(x\right)\right) = 0$$

Assignment
Page 63 (AP Calc Text)
#'s 43,44
Handout 2.1
Limits of Combined and
Composite Functions

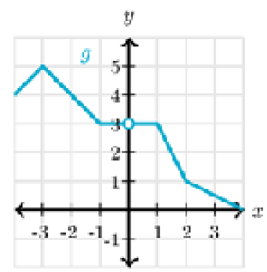
2.1 Limits of Combined and Composite Functions

Functions f and g are graphed.



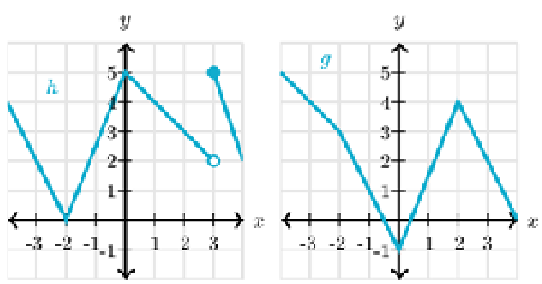
Find $\lim\limits_{x \to 2} \left(f(x) - g(x) \right)$.

Function g is graphed.



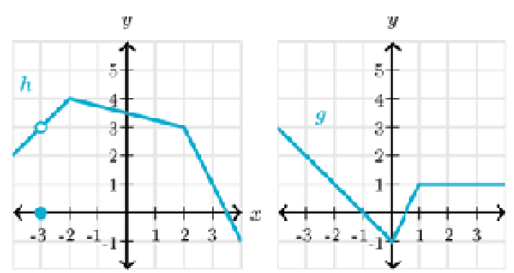
Find $\lim\limits_{x \to 0} \left(-2g(x) \right)$.

Functions h and g are graphed.



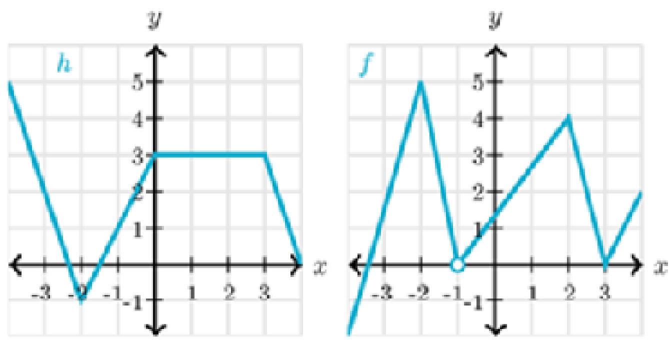
Find $\lim\limits_{x \to 3} \left(h(x) - g(x) \right)$.

Functions h and g are graphed.



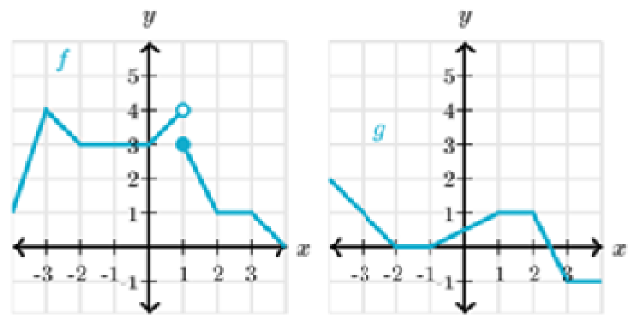
Find $\lim\limits_{x \to -3} \left(h(x) + g(x) \right)$.

Functions h and f are graphed.



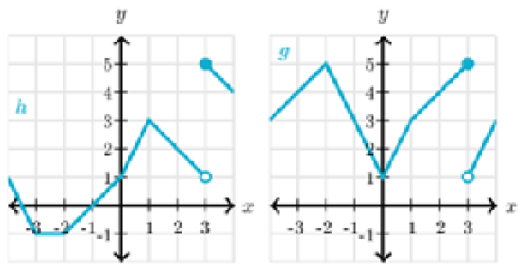
Find $\lim_{x\to -1} \frac{h(x)}{f(x)}$.

Functions f and g are graphed.



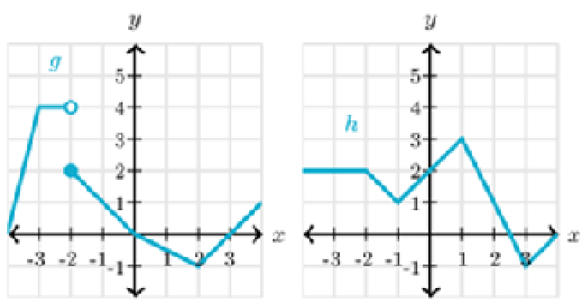
Find $\lim_{x\to 1} \frac{f(x)}{g(x)}$.

Functions h and g are graphed.



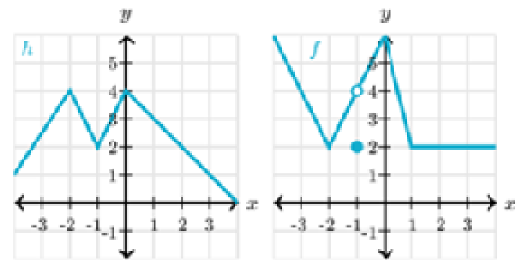
Find $\lim\limits_{x\to 3}\Big(h(x)g(x)\Big)$.

Functions g and h are graphed.



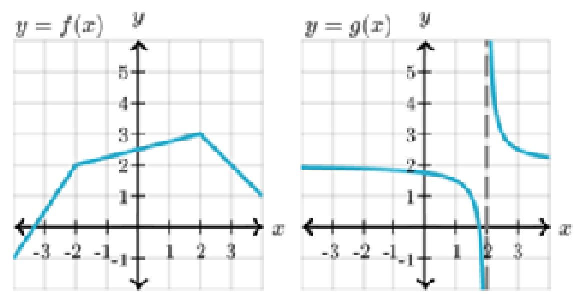
Find $\lim\limits_{x\to -2}\Big(g(x)h(x)\Big)$.

Functions h and f are graphed.



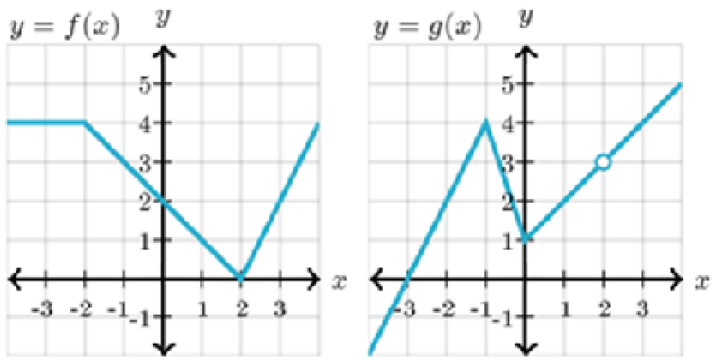
Find $\lim\limits_{x\to -1}\Big(h(x)f(x)\Big)$.

Functions f and g are graphed.



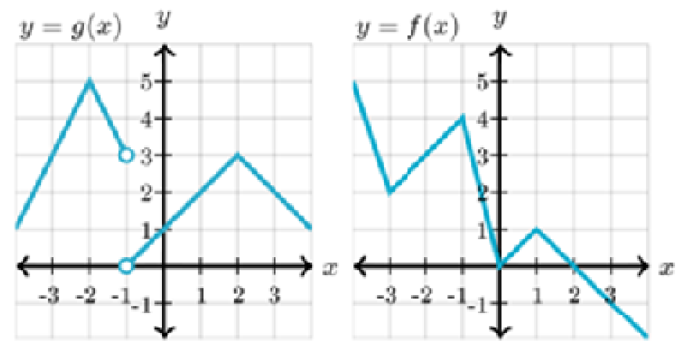
Find $\lim\limits_{x\to 2} f\Big(g(x)\Big)$.

Functions f and g are graphed.



Find $\lim_{x \to 2} f\Big(g(x)\Big)$.

Functions g and f are graphed.



Find $\lim_{x \to 3} g\Big(f(x)\Big)$.