# 11.3 The Binomial Theorem

In this section we will learn a more efficient way to expand binomials.

If we expanded the following binomials we would get the following:

| Binomial | Pascal's Triangle in Binomial Expansion | Row |
|---|---|---|
| $(x+y)^0$ | $1$ | 1 |
| $(x+y)^1$ | $1x + 1y$ | 2 |
| $(x+y)^2$ | $1x^2 + 2xy + 1y^2$ | 3 |
| $(x+y)^3$ | $1x^3 + 3x^2y + 3xy^2 + 1y^3$ | 4 |
| $(x+y)^4$ | $1x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + 1y^4$ | 5 |

How does the row # of Pascal's triangle relate to the exponent in the binomial?

row # is 1 greater than exponent

How does the # of terms in each row relate to the exponent of the binomial?

# terms is one more than exponent

The coefficients in a binomial expansion can also be determined using combinations.

| Pascal's Triangle | Combinations |
|---|---|
| 1 | ${}_0C_0$ |
| 1 1 | ${}_1C_0$ ${}_1C_1$ |
| 1 2 1 | ${}_2C_0$ ${}_2C_1$ ${}_2C_2$ |
| 1 3 3 1 | ${}_3C_0$ ${}_3C_1$ ${}_3C_2$ ${}_3C_3$ |
| 1 4 6 4 1 | ${}_4C_0$ ${}_4C_1$ ${}_4C_2$ ${}_4C_3$ ${}_4C_4$ |
| 1 5 10 10 5 1 | ${}_5C_0$ ${}_5C_1$ ${}_5C_2$ ${}_5C_3$ ${}_5C_4$ ${}_5C_5$ |

1 6 15 20 15 6 1

Example 1: Expand $(x + y)^5$ by either using Pascal's triangle or combinations.

expansion has 6 terms.

$$({}_5C_0)x^5y^0 + {}_5C_1x^4y^1 + {}_5C_2x^3y^2 + {}_5C_3x^2y^3$$

$$+ {}_5C_4xy^4 + {}_5C_5x^0y^5$$

$$= (1)(x^5)(1) + 5x^4y^1 + 10x^3y^2 + 10x^2y^3$$

$$+ 5xy^4 + (1)(1)y^5$$

$$= x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$$

$$t_1 \qquad t_2 \qquad t_3 \qquad t_4 \qquad t_5 \qquad t_6$$

You can use the **binomial theorem** to expand any power of a binomial expression.

$$(x + y)^n = {}_nC_0(x)^n(y)^0 + {}_nC_1(x)^{n-1}(y)^1 + {}_nC_2(x)^{n-2}(y)^2 + \cdots + {}_nC_{n-1}(x)^1(y)^{n-1} + {}_nC_n(x)^0(y)^n$$

The following are some important observations about the expansion of $(x + y)^n$, where $x$ and $y$ represent the terms of the binomial and $n \in \mathrm{N}$:

- the expansion contains $n + 1$ terms
- the number of objects, $k$, selected in the combination ${}_nC_k$ can be taken to match the number of factors of the second variable selected; that is, it is the same as the exponent on the second variable
- the general term, $t_{k+1}$, has the form

$${}_nC_k(x)^{n-k}(y)^k$$

the same

- the sum of the exponents in any term of the expansion is $n$

Example 2: Expand $(2a - 3b)^4$

$${}_4C_0(2a)^4 + {}_4C_1(2a)^3(-3b)^1 + {}_4C_2(2a)^2(-3b)^2$$

$$+ {}_4C_3(2a)^1(-3b)^3 + {}_4C_4(-3b)^4$$

$$= 16a^4 + 4(8a^3)(-3b) + 6(4a^2)(9b^2)$$

$$+ 4(2a)(-27b^3) + 1(81b^4)$$

$$= 16a^4 - 96a^3b + 216a^2b^2 - 216ab^3 + 81b^4$$

Example 3: What is the fourth term in the expansion $(3a-5)^5$

${}_5C_0 \quad {}_5C_1 \quad {}_5C_2 \quad {}_5C_3$

$$= {}_5C_3 (3a)^{2} (-5)^{3}$$

$$= 10(9a^2)(-125)$$

$$= -11250a^2$$

## Your Turn

**a)** How many terms are in the expansion of $(2a - 7)^8$?

**b)** What is the value of the fourth term in the expansion of $(2a - 7)^8$?

**c)** Use the binomial theorem to find the first four terms of the expansion of $(3a + 2b)^7$.

a) 9

b) ${}_8C_3(2a)^5(-7)^3$

$= 56(32a^5)(-343)$

$= -614656a^5$

$(3a+2b)^7$

c) ${}_7C_0(3a)^7 + {}_7C_1(3a)^6(2b)^1 + {}_7C_2(3a)^5(2b)^2 + {}_7C_3(3a)^4(2b)^3$

$= (1)(2187a^7) + 7(729a^6)(2b) + 21(243a^5)(4b^2) + 35(81a^4)(8b^3)$

$= 2187a^7 + 12393a^6 b + 20412a^5b^2 + 22680a^4b^3$

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#'s 4, 5, 6, 7, 11, 17, 20